Applying stochastic time changes to Lévy processes

Liuren Wu

Zicklin School of Business, Baruch College

Option Pricing
Outline

1. Stochastic time change

2. Option pricing

3. Model Design
Outline

1. Stochastic time change

2. Option pricing

3. Model Design
Lévy processes can generate different iid return innovation distributions.

- Any distribution you can think of, we can specify a Lévy process, with the increments of the process matching that distribution.
- Caveat: The same type of distribution applies to all time horizons — you may not be able to specify the distribution simultaneously at different time horizons.

Lévy processes cannot generate distributions that vary over time.

- Returns modeled by Lévy processes generate option implied volatility surfaces that stay the same over time (as a function of standardized moneyness and time to maturity).

Lévy processes cannot capture the following salient features of the data:

- Stochastic volatility
- Stochastic risk reversal (skewness)
- Stochastic correlation
Capturing stochastic volatility via time changes

- Discrete-time analog again: $R_{t+1} = \mu_t + \sigma_t \varepsilon_{t+1}$
  - $\varepsilon_{t+1}$ is an iid return innovation, with an arbitrary distribution assumption $\leftrightarrow$ Lévy process.
  - $\sigma_t$ is the conditional volatility, $\mu_t$ is the conditional mean return, both of which can be time-varying, stochastic...

- In continuous time, how do we model stochastic mean/volatility tractably?
  - If the return innovation is modeled by a Brownian motion, we can let the instantaneous variance to be stochastic and tractable, not volatility (Heston(1993), Bates (1996)).
  - If the return innovation is modeled by a compound Poisson process, we can let the Poisson arrival rate to be stochastic, not the mean jump size, jump distribution variance (Bates(2000), Pan(2002)).

- If the return innovation is modeled by a general Lévy process, it is tractable to randomize the time, or something proportional to time.

Variance of a Brownian motion, intensity of a Poisson process are both proportional to time.
Randomize the time

- Review the Lévy-Khintchine Theorem:

\[
\phi(u) \equiv \mathbb{E}[e^{iuX_t}] = e^{-t\psi(u)}, \\
\psi(u) = -iu\mu + \frac{1}{2}u^2\sigma^2 \quad \text{for diffusion with drift } \mu \text{ and variance } \sigma^2, \\
\psi(u) = \lambda \left(1 - e^{iu\mu J - \frac{1}{2}u^2v_J}\right) \quad \text{for Merton’s compound Poisson jump.}
\]

- The drift \( \mu \), the diffusion variance \( \sigma^2 \), and the Poisson arrival rate \( \lambda \) are all proportional to time \( t \).

- We can directly specify \( (\mu_t, \sigma^2_t, \lambda_t) \) as following stochastic processes.

- Or we can randomize time \( t \rightarrow T_t \) for the same result.

- We define \( T_t \equiv \int_0^t \nu_s \, ds \) as the (stochastic) \textit{time change}, with \( \nu_t \) being the \textit{instantaneous activity rate}.

  - Depending on the Lévy specification, the activity rate has the same meaning (up to a scale) as a randomized version of the \textit{instantaneous drift}, \textit{instantaneous variance}, or \textit{instantaneous arrival rate}. 
Applying separate time changes

... to different Lévy components

- Consider a Lévy process $X_t \sim (\mu, \sigma^2, \lambda p(x))$.
  - If we apply random time change to $X_t \rightarrow X_{T_t}$ with $T_t = \int_0^t \nu_s ds$, it is equivalent to assuming that $(\mu_t, \sigma_t^2, \lambda_t)$ are all time varying, but they are all proportional to one common source of variation $\nu_t$.
  - If $(\mu_t, \sigma_t^2, \lambda_t)$ vary separately, we need to apply separate time changes to the three Lévy components.
    - Decompose $X_t$ into three Lévy processes: $X_t^1 \sim (\mu, 0, 0)$, $X_t^2 \sim (0, \sigma^2, 0)$, and $X_t^1 \sim (0, 0, \lambda p(x))$, and then apply separate time changes to the three Lévy processes.
Interpretation I

- We can think of $t$ as the calendar time, and $T_t$ as the business time.

- Business activity accumulates with calendar time, but the speed varies, depending on the business activity.

- At heavy trading hours, one hour on a clock generates two hours worth of business activity ($v_t = 2$).

- At afterhours, one hour generates half hour of activity ($v_t = 1/2$).

- Business activity tends to intensify before earnings announcements, FOMC meeting days...

- In this sense, $v_t$ captures the intensity of business activity at a certain time $t$.

- Ané & Geman (2000, JFE): Stock returns are not normally distributed, but become normally distributed when they are scaled by number of trades.
We use Lévy processes to model return innovations and stochastic time changes to generate stochastic volatility and higher moments...

We can think of each Lévy process as capturing one source of economic shock.

The stochastic time change on each Lévy process captures the random intensity of the impact of the economic shock on the financial security.

\[
\text{Return } \sim \sum_{i=1}^{K} X_{T_i}^j \sim \sum_{i=1}^{K} (\text{Economic Shock From Source } i)_{\text{Stochastic impacts}}.
\]

I like this interpretations. Many economic phenomena can be modeled in terms of economic shocks and (time varying) intensities.
In 1949, Bochner introduced the notion of time change to stochastic processes. In 1973, Clark suggested that time-changed diffusions could be used to accurately describe financial time series.

At present, there are two types of clocks used to model business time:

1. Continuous clocks have the property that business time is always strictly increasing over calendar time.
2. Clocks based on increasing jump processes have staircase like paths.

The first type of business clock can be used to describe stochastic volatility — which is what we do in this section.

The second type of clock can transform a diffusion into a jump process — All Lévy processes considered in the previous section can be generated as changing the clock of a diffusion with an increasing jump process (subordinator).

All semimartingales can be written as time-changed Brownian motion.
1. Stochastic time change

2. Option pricing

3. Model Design
To compute the time-0 price of a European option price with maturity at \( t \), we first compute the Fourier transform of the log return \( \ln S_t / S_0 \). Then we compute option value via Fourier inversions.

The Fourier transform of a time-changed Lévy process:

\[
\phi_Y(u) \equiv \mathbb{E}^Q \left[ e^{iuX_{T_t}} \right] = \mathbb{E}^Q \left[ e^{iuX_{T_t} + \psi_x(u)T_t} e^{-\psi_x(u)T_t} \right] = \mathbb{E}^M \left[ e^{-\psi_x(u)T_t} \right], \quad u \in D \subset \mathbb{C},
\]

where the new measure \( M \) is defined by the exponential martingale:

\[
\frac{dM}{dQ}\bigg|_t = \exp \left( iuX_{T_t} + T_t \psi_x(u) \right).
\]

- Without time-change, \( e^{iuX_t + t\psi_x(u)} \) is an exponential martingale by Lévy-Khintchine Theorem.

- A continuous time change does not change the martingality.


- \( M \) is complex valued (no longer a probability measure).
When $X$ and $\mathcal{T}_t = \int_0^t \nu_s ds$ are independent, we have

$$
\phi_Y(u) \equiv \mathbb{E}_Q \left[ e^{iuX_{\mathcal{T}_t}} \right] = \mathbb{E}_Q \left[ \mathbb{E}_Q \left[ e^{iuX_{\mathcal{T}_t}} \mid \mathcal{T}_t = Z \right] \right] \\
= \mathbb{E}_Q \left[ e^{-\psi(x)(u)\mathcal{T}_t} \right],
$$

by law of iterated expectations.

- No measure change is necessary.
- The operation is similar to Hull and White (1987): The option value of an independent stochastic volatility model is written as the expectation of the BMS formula over the distribution of the integrated variance, $\mathcal{T}_t = \int_0^t \nu_s ds$.
- A continuous time change does not change the martingality.
- When $X$ and $\nu_t$ are correlated, the measure change from $\mathbb{Q}$ to $M$ hides the correlation under the new measure.
- If we must give a name, let’s call $M$ the **correlation neutral measure**.
The Fourier transform of a time-changed Lévy process:

\[ \phi_Y(u) \equiv \mathbb{E}^Q \left[ e^{iuX_T} \right] = \mathbb{E}^M \left[ e^{-\psi_X(u)T} \right] \]

Tractability of the transform \( \phi(u) \) depends on the tractability of:

1. The characteristic exponent of the Lévy process \( \psi_X(u) \)
   - Tractable Lévy specifications include: Brownian motion, (Compound) Poisson, DPL, NIG, ... (done in previous section)

2. The Laplace transform of \( T_t \) under \( M \).
   - Tractable Laplace comes from activity rate dynamics: affine, quadratic, Wishart, 3/2
   - The measure change from \( Q \) to \( M \) is defined by an exponential martingale.

The two \( (X, T_t) \) can be chosen separately as building blocks, for different purposes.
We have solved the characteristic exponent of the Lévy process (by the Lévy-Khintchine Theorem).

Now we try to solve the Laplace transform of the stochastic time,

\[ \mathcal{L}_T(\psi) \equiv \mathbb{E} \left[ e^{-\psi T_t} \right] = \mathbb{E} \left[ e^{-\psi \int_0^t v_s ds} \right] \] (1)

Recall the pricing equation for zero-coupon bonds:

\[ B(0, t) \equiv \mathbb{E}^Q \left[ e^{-\int_0^t r_s ds} \right] \] (2)

The two pricing equations look analogous (even though they are not related)

- Both \( v_t \) and \( r_t \) need to be positive.
- If we set \( r_t = \psi v_t \), \( \mathcal{L}_T(\psi) \) is essentially the bond price.

The similarity allows us to borrow the vast literature on bond pricing:

- **Affine class**: Zero-coupon bond prices are exponential affine in the state variable.
- **Quadratic**: Zero-coupon bond prices are exponential quadratic in the state variable.

...
A long list of papers propose different dynamic term structure models:

- **Specific examples:**
  - Vasicek, 1977, JFE: The instantaneous interest rate follows an Ornstein-Uhlenbeck process.
  - Cox, Ingersoll, Ross, 1985, Econometrica: The instantaneous interest rate follows a square-root process.
  - Many multi-factor examples ...

- **Classifications (back-filling)**
  - Duffie, Kan, 1996, Mathematical Finance: Spot rates are affine functions of state variables.
  - Leippold, Wu, 2002, JFQA: Spot rates are quadratic functions of state variables.
  - Filipovic, 2002, Mathematical Finance: How far can we go?
  - Gabaix: Bond prices are affine.
Identifying dynamic term structure models:
The forward and backward procedures

- The traditional procedure:
  - First, we make assumptions on factor dynamics ($Z$), market prices ($\gamma$), and how interest rates are related to the factors $r(Z)$, based on what we think is reasonable.
  - Then, we derive the fair valuation of bonds based on these dynamics and market price specifications.

- The back-filling (reverse engineering) procedure:
  - First, state the form of solution that we want for bond prices.
  - Then, figure out what dynamics specifications generate the pricing solutions that we want.
    - The dynamics are not specified to be reasonable, but specified to generate a form of solution that we like.

- It is good to be able to go both ways.
  - It is important not only to understand existing models, but also to derive new models that meet your work requirements.
What we want: Zero-coupon bond prices are exponential affine functions of state variables.
- Continuously compounded spot rates are affine in state variables.
- It is simple and tractable. We can use spot rates as factors.

Let $Z$ denote the state variables, let $B(Z_t, \tau)$ denote the time-$t$ fair value of a zero-coupon bond with time to maturity $\tau = T - t$, we have

$$B(Z_t, \tau) = \mathbb{E}_t^Q \left[ \exp \left( - \int_t^\tau r(Z_s) \, ds \right) \right] = \exp \left( -a(\tau) - b(\tau)^\top Z_t \right)$$

By writing $B(Z_t, \tau)$ and $r(Z_t)$, and solutions $a(\tau), b(\tau)$, I am implicitly focusing on time-homogeneous models. Calendar dates do not matter. This assumption is for (notational) simplicity more than anything else.

With calendar time dependence, the notation can be changed to, $B(Z_t, t, T)$ and $r(Z_t, t)$. The solutions would be $a(t, T), b(t, T)$.

Questions to be answered:
- What is the short rate function $r(Z_t)$?
- What’s the dynamics of $Z_t$ under measure $Q$?
Diffusion dynamics

- To make the derivation easier, let’s focus on diffusion factor dynamics:
  \[ dZ_t = \mu(Z)dt + \sigma(Z)dW_t \text{ under } \mathbb{Q}. \]

- We want to know: What kind of specifications for \( \mu(Z), \sigma(Z) \) and \( r(Z) \) generate the affine solutions?

- For a generic valuation problem,
  \[
  f(Z_t, t, T) = \mathbb{E}_t^\mathbb{Q} \left[ \exp \left( - \int_t^T r(Z_s)ds \right) \Pi_T \right],
  \]
  where \( \Pi_T \) denotes terminal payoff, the value satisfies the following partial differential equation:
  \[
  f_t + \mathcal{L}f = rf, \quad \mathcal{L}f - \text{infinitesimal generator}
  \]
  with boundary condition \( f(T) = \Pi_T \).

- Apply the PDE to the bond valuation problem,
  \[
  B_t + B_Z^\top \mu(Z) + \frac{1}{2} \sum B_{ZZ} \cdot \sigma(Z)\sigma(Z)^\top = rB
  \]
  with boundary condition \( B(Z_T, 0) = 1 \).
Starting with the PDE,

\[ B_t + B_Z^T \mu(Z) + \frac{1}{2} \sum B_{ZZ} \cdot \sigma(Z)\sigma(Z)^T = rB, \quad B(Z_T, 0) = 1. \]

If \( B(Z_t, \tau) = \exp(-a(\tau) - b(\tau)^T Z_t) \), we have

\[
B_t = B \left( a'(\tau) + b'(\tau)^T Z_t \right), \quad B_Z = -Bb(\tau), \quad B_{ZZ} = Bb(\tau)b(\tau)^T,
\]

\[
y(t, \tau) = \frac{1}{\tau} \left( a(\tau) + b(\tau)^T Z_t \right), \quad r(Z_t) = a'(0) + b'(0)^T Z_t = a_r + b_r^T Z_t.
\]

Plug these back to the PDE,

\[
a'(\tau) + b'(\tau)^T Z_t - b(\tau)^T \mu(Z) + \frac{1}{2} \sum b(\tau)b(\tau)^T \cdot \sigma(Z)\sigma(Z)^T = a_r + b_r^T Z_t
\]

Question: What specifications of \( \mu(Z) \) and \( \sigma(Z) \) guarantee the above PDE to hold at all \( Z \)?

- Power expand \( \mu(Z) \) and \( \sigma(Z)\sigma(Z)^T \) around \( Z \) and then collect coefficients of \( Z^p \) for \( p = 0, 1, 2 \cdots \). These coefficients have to be zero separately for the PDE to hold at all times.
Back filling

\[ a'(\tau) + b'(\tau)^T Z_t - b(\tau)^T \mu(Z) + \frac{1}{2} \sum b(\tau)b(\tau)^T \cdot \sigma(Z)\sigma(Z)^T = a_r + b_r^T Z_t \]

- Set \( \mu(Z) = a_m + b_m Z + c_m ZZ^T + \cdots \) and 
  \[ \left[ \sigma(Z)\sigma(Z)^T \right]_i = \alpha_i + \beta_i^T Z + \eta_i ZZ^T + \cdots, \] 
  and collect terms:
  
  \[
  \begin{align*}
  \text{constant} & \quad a'(\tau) - b(\tau)^T a_m + \frac{1}{2} \sum b(\tau)b(\tau)^T \cdot \alpha_i = a_r \\
  \text{Z} & \quad b'(\tau)^T - b(\tau)^T b_m + \frac{1}{2} \sum b(\tau)b(\tau)^T \cdot \beta_i^T = b_r^T \\
  \text{ZZ}^T & \quad -b(\tau)^T c_m + \frac{1}{2} \sum b(\tau)b(\tau)^T \cdot \eta_i = 0
  \end{align*}
  \]

- The quadratic and higher-order terms are almost surely zero.

- We thus have the conditions to have exponential-affine bond prices:
  \[ \mu(Z) = a_m + b_m Z, \quad \left[ \sigma(Z)\sigma(Z)^T \right]_i = \alpha_i + \beta_i^T Z, \quad r(Z) = a_r + b_r^T Z. \]

- We can solve the coefficients \([a(\tau), b(\tau)]\) via the following ordinary differential equations:
  \[
  \begin{align*}
  a'(\tau) &= a_r + b(\tau)^T a_m - \frac{1}{2} \sum b(\tau)b(\tau)^T \cdot \alpha_i \\
  b'(\tau) &= b_r + b_m^T b(\tau) - \frac{1}{2} \sum b(\tau)b(\tau)^T \cdot \beta_i
  \end{align*}
  \]

  starting at \( a(0) = 0 \) and \( b(0) = 0 \).
Quadratic and others

- Add jumps to the affine dynamics: The arrival rate of jumps need to be affine in the state vector.
- Can you identify the conditions for quadratic models: Bond prices are exponential quadratic in state variables?
- Can you identify the conditions for “cubic” models: Bond prices are exponential cubic in state variables?
- Affine bond prices: Recently Xavier Gabaix derive a model where bond prices are affine (not exponential affine!) in state variables.
From affine DTSM to affine activity rates

- Review of affine DTSM: \( B(Z_0, t) \equiv \mathbb{E} \left[ e^{-\int_0^t r_s ds} \right] = e^{-a(t)-b(t)^\top Z_0} \) if

\[ r_t = a_r + b_r^\top Z_t, \mu(Z) = \kappa(\theta - Z), [\sigma(Z)\sigma(Z)^\top]_{ii} = \alpha_i + \beta_i^\top Z \]

- By analogy, if we want:

\[ \mathcal{L}_T(z) \equiv \mathbb{E} \left[ e^{-z T_t} \right] = \mathbb{E} \left[ e^{-z \int_0^t \nu_s ds} \right] = e^{-a(t)-b(t)^\top Z_0}, \text{ we can set} \]

\[ \nu_t = a_\nu + b_\nu^\top Z_t, \mu(Z) = \kappa(\theta - Z), [\sigma(Z)\sigma(Z)^\top]_{ii} = \alpha_i + \beta_i^\top Z \]

- Problem: We need the affine dynamics under the complex-valued measure \( \mathbb{M} \). Correlations between the Lévy process \( X \) and the state vector \( Z \) can make the whole thing messy. Affine dynamics under \( \mathbb{Q} \) are no guarantee for exponential affine solution.

- We use some concrete examples to show how this works.
Example: The Heston (1996) stochastic volatility model

- SDE: $dS_t/S_t = (r - q)dt + \sqrt{v_t}dW_t$ with $dv_t = \kappa(\theta - v_t)dt + \sigma\sqrt{v_t}dW^\nu_t$, $\mathbb{E}[dW_t dW^\nu_t] = \rho dt$.

- We can write the security return as a time-changed Lévy process,

$$\ln S_t/S_0 = (r - q)t + W_{T_t} - \frac{1}{2} T_t, \quad T_t = \int_0^t v_s ds.$$ 

- The Fourier transform of the return,

$$\phi_s(u) \equiv \mathbb{E}^Q \left[ e^{iu \ln S_t / S_0} \right] = e^{iu(r-q)t} \mathbb{E}^Q \left[ e^{iu(W_{T_t} - \frac{1}{2} T_t)} \right]$$

$$= e^{iu(r-q)t} \mathbb{E}^Q \left[ e^{iu(W_{T_t} - \frac{1}{2} T_t) + \psi(u) T_t - \psi(u) T_t} \right]$$

$$= e^{iu(r-q)t} \mathbb{E}^M \left[ e^{-\psi(u) T_t} \right]$$

where $\psi(u) = \frac{1}{2}(iu + u^2)$ is the characteristic exponent of the Lévy process $(W_t - \frac{1}{2} t)$.

- To solve the Laplace transform, we need the dynamics of $v$ under $\mathbb{M}$. 

Liuren Wu (Baruch)  Stochastic time changes  Option Pricing  24 / 38
Example: Heston

- SDE: \( \frac{dS_t}{S_t} = (r - q)dt + \sqrt{\nu_t}dW_t \) with 
  \[ dv_t = \kappa(\theta - \nu_t)dt + \sigma_v \sqrt{\nu_t}dW_t^\nu, \quad \mathbb{E}[dW_t dW_t^\nu] = \rho dt. \]

- The measure change: \( \frac{d\mathbb{M}}{d\mathbb{Q}} = \exp(iu(W_{T_t} - \frac{1}{2}T_t) + \psi(u)T_t) \).

- The \( \nu \) dynamics under \( \mathbb{M} \):
  \[
  dv_t = \kappa(\theta - \nu_t)dt + \mathbb{E}[iudW_{T_t}\sigma_v \sqrt{\nu_t}dW_t^\nu] + \sigma_v \sqrt{\nu_t}dW_t^\nu \\
  = \kappa(\theta - \nu_t)dt + iu\sigma_v \rho \nu_t dt + \sigma_v \sqrt{\nu_t}dW_t^\nu \\
  = (\kappa \theta - \kappa^\mathbb{M} \nu_t)dt + \sigma_v \sqrt{\nu_t}dW_t^\nu
  
  \]
  with \( \kappa^\mathbb{M} = \kappa - iu\sigma_v \rho \).

- Note that \( dW_{T_t} \) and \( \sqrt{\nu_t}dW_t \) are equivalent in distribution.

- Since the \( \nu \) dynamics are affine under \( \mathbb{M} \), we have the Laplace transform exponential affine in \( \nu \),
  \[
  \phi_s(u) = \mathbb{E}^\mathbb{Q} \left[ e^{iu \ln S_t/S_0} \right] = e^{iu(r-q)t} \mathbb{E}^\mathbb{Q} \left[ e^{iu(W_{T_t} - \frac{1}{2}T_t)} \right] \\
  = e^{iu(r-q)t} \mathbb{E}^\mathbb{M} \left[ e^{-\psi(u)T_t} \right] = e^{iu(r-q)t - a(t) - b(t)v_0}
  
  \]
  with \( a'(t) = b(t)\kappa \theta \) and \( b'(t) = \psi(u) - \kappa^\mathbb{M} b(t) - \frac{1}{2} b(t)^2 \sigma_v^2 \), starting at \( a(0) = b(0) = 0 \).
Example: Heston—Not all affine works

- The key for tractability is to maintain the \( v \) dynamics affine under \( \mathbb{M} \).

- Suppose we extend the Heston specification:
  \[
  d v_t = \kappa (\theta - v_t) dt + \sigma_v \sqrt{\alpha + \beta v_t} d W_t^v,
  \]
  which is still affine under \( \mathbb{Q} \), but it is no longer affine under \( \mathbb{M} \):
  \[
  d v_t = \kappa (\theta - v_t) dt + \mathbb{E}[i u d W_{T_t}, \sigma_v \sqrt{\alpha + \beta v_t} d W_t^v] + \sigma_v \sqrt{v_t} d W_t^v
  \]
  unless we redefine the time change as \( T_t = \int_0^t (\alpha + \beta v_s) ds \) or we set \( \rho = 0 \).

- The constant volatility specification does not work either:
  \[
  d v_t = \kappa (\theta - v_t) dt + \sigma_v d W_t^v
  \]
  (in addition to the fact that \( v_t \) can go to zero this time).

- A general affine specification \( v_t = a_v + b_v^\top Z_t \) does not always work for the same reason, but the following two-factor specification works:
  \[
  d v_t = \kappa (m_t - v_t) dt + \sigma_v \sqrt{v_t} d W_t^v, \quad d m_t = \kappa_m (\theta_m - m_t) dt + \sigma_m \sqrt{m_t} d W_t^m
  \]
  with \( \mathbb{E}[d W_t^m d W_t^v] = \mathbb{E}[d W_t^m d W_t] = 0 \).
Example: Bates jump-diffusion stochastic volatility model

- SDE: \( dS_t/S_t = (r - q)dt + \sqrt{v_t}dW_t + dJ(\lambda) - \lambda(e^{\mu_J + \frac{1}{2}\nu_J} - 1)dt \) with \( dv_t = \kappa(\theta - v_t)dt + \sigma_v\sqrt{v_t}dW^v_t, \mathbb{E}[dW_t dW^v_t] = \rho dt \).
  The stock price process includes compound Poisson jump process, with arrival rate \( \lambda \).
  Conditional on a jump occurring, the jump size in return has a normal distribution \((\mu_J, \nu_J)\).

- We can write the security return as a time-changed Lévy process,
  \[ \ln S_t/S_0 = (r - q)t + [W_{\mathcal{T}_t} - \frac{1}{2} \mathcal{T}_t] + [X_t - k_x(1)t], \quad \mathcal{T}_t = \int_0^t v_s ds. \]
  where \( X \) denotes a pure-jump Lévy process with Lévy density given by
  \[ \pi(x) = \lambda \frac{1}{\sqrt{2\pi\nu_J}} e^{-\frac{(x-\mu_J)^2}{2\nu_J}}. \]
  The cumulant exponent is
  \[ k_x(s) = \int_{\mathbb{R}_0} (e^{sx} - 1)\pi(x)dx = \lambda \left(e^{s\mu_J + \frac{1}{2}s^2\nu_J} - 1 \right). \]
The Fourier transform of the return, 
\[ \phi_s(u) \equiv \mathbb{E}^Q \left[ e^{iu \ln S_t / S_0} \right] = e^{iu(r-q)t} \mathbb{E}^Q \left[ e^{iu \left( W_{T_t} - \frac{1}{2} T_t \right)} \right] \mathbb{E}^Q \left[ e^{iu \left( X_{T_t} - k_X(1)t \right)} \right] \]

where \([a(t), b(t)]\) come directly from the Heston model and the characteristic exponent of the pure-jump Lévy process (concavity adjusted) is:
\[
\psi_J(u) = \int_{\mathbb{R}_0} (1-e^{iux})\pi(x)dx + iuk_X(1) = \lambda (1-e^{iu\mu_J - \frac{1}{2} u^2 v_J}) + iu\lambda \left( e^{\mu_J + \frac{1}{2} v_J} - 1 \right)
\]

By definition, jumps are orthogonal to diffusion. Hence, the two components can be processed separately.

Question: Why just time change diffusion \( W \)? Why not also time change the jump \( X \)?

\[
\ln S_t / S_0 = (r-q)t + [W_{T_t} - \frac{1}{2} T_t] + [X_{T_t} - k_X(1)T_t^x].
\]

Also, replace the compound Possion jump with any type of jump you like — SV4 in Huang and Wu (2004).
1. Stochastic time change

2. Option pricing

3. Model Design
Model design: General principles

- Start with the risk-neutral ($\mathbb{Q}$) process — That’s where tractability is needed the most dearly.
  - Identify the economic sources, model each with a Lévy process ($X^k_t$).
  - Decide whether to apply separate time changes: $X^k_t \rightarrow X^k_{T^k_t}$ to make the impact of this economic source stochastic over time.
  - Concavity adjust each component to guarantee the martingale condition: $\mathbb{E}^\mathbb{Q}[S_t/S_0] = e^{(r-q)t}$.

\[
\ln \frac{S_t}{S_0} = (r - q)t + \sum_{k=1}^{K} \left( b^k X^k_{T^k_t} - k_x(b^k) T^k_t \right),
\]

- For tractability,
  - Use Lévy processes that generate tractable characteristic exponents ($\psi_X(u)$).
  - Use time changes that generate tractable Laplace transforms
  - Orthogonalize the economic shocks $X^k$ such that

\[
\phi_s(u) = e^{iu(r-q)t} \prod_{k} \mathbb{E} \left[ e^{iu \left( b^k X^k_{T^k_t} - k_x(b^k) T^k_t \right)} \right]
\]
Market prices and statistics dynamics

- Since we can always use Euler approximation for model estimation, tractability requirement is not as strong for the statistical dynamics.

- We can specify pretty much any forms for the market prices subject to (i) technical conditions, (ii) economic sensibility, and (iii) identification concerns.

- Simple/parsimonious specification: Constant market prices of return and vol risks \((\gamma_k, \gamma_{kv})\)

\[
\mathcal{M}_t = e^{-rt} \prod_{k=1}^{K} \exp \left( -\gamma_k X_{T_t}^k - \phi_k (\gamma_k) T_t^k - \gamma_{kv} X_{T_t}^{kv} - \phi_{kv} (\gamma_{kv}) T_t^k \right) \cdot \zeta,
\]

- \(\sigma W_t \rightarrow \) constant drift adjustment \(\eta = \gamma \sigma^2\).

- Pure jump Lévy process \(\rightarrow \pi^P(x) = e^{\gamma x} \pi^Q(x)\), drift adjustment:
  \[
  \eta = \varphi^P_j(1) - \varphi^Q_j(1) = \varphi^Q_j(1 + \gamma) - \varphi^Q_j(\gamma) - \varphi^Q_j(1).
  \]

- Time change: instantaneous risk premium \((\eta \nu_t)\) proportional to the risk level \(\nu_t\).
Example: Return on a stock

- Model the return on a stock to reflect shocks from **two sources**:
  - **Credit risk**: In case of corporate default, the stock price falls to zero. Model the impact as a Poisson Lévy jump process with log return jumps to negative infinity upon jump arrival.
  - **Market risk**: Daily market movements (small or large). Model the impact as a diffusion or infinite-activity (infinite variation) Lévy jump process or both.

- Apply separate time changes to the two Lévy components to capture (1) the intensity variation of corporate default, (2) the market risk (volatility) variation.

- **Key**: *Each component has a specific economic purpose.*

Example: A CAPM model:

\[
\ln \frac{S_t^i}{S_0^i} = (r - q)t + \left( \beta^j X_t^m - \varphi_x^m(\beta^j) T_t^m \right) + \left( X_{T_t}^j - \varphi_x(1) T_t^j \right).
\]

- Estimate \( \beta \) and market prices of return and volatility risk using index and single name options.
- Cross-sectional analysis of the estimates:
  Are the beta estimates similar to estimates from time-series stock return regressions?

An international CAPM:

Exchange rate reflects the interaction between two economic forces.

Use two Lévy processes to model the two economic forces separately.

Consider a negatively skewed distribution (downside jumps) from each economic source (crash-o-phobia from both sides). Use the difference to model the currency return between the two economies.

Apply separate time changes to the two Lévy processes to capture the strength variation of the two economic forces.

Stochastic time changes on the two negatively skewed Lévy processes generate both stochastic volatility and stochastic skew.

Key: Each component has its specific economic purpose.

Dollar price of peso drops by a significant amount when Mexico defaults on its sovereign debt.

- Currency return on peso contains both market risk and credit risk.
- The intensities of both types of risks are stochastic (and probably correlated).

Exchange rates and pricing kernels

- Exchange rate reflects the interaction between two economic forces.
- The economic meaning becomes clearer if we model the pricing kernel of each economy.
  - Let $m_{0,t}^{US}$ and $m_{0,t}^{JP}$ denote the pricing kernels of the US and Japan. Then the dollar price of yen $S_t$ is given by
    \[
    \ln \frac{S_t}{S_0} = \ln m_{0,t}^{JP} - \ln m_{0,t}^{US}.
    \]
  - If we model the negative of the logarithm of each pricing kernel ($-\ln m_{0,t}^j$) as a time-changed Levy process, $X_{T_t}^j$ ($j = US, JP$) with negative skewness. Then, $\ln \frac{S_t}{S_0} = \ln m_{0,t}^{JP} - \ln m_{0,t}^{US} = X_{T_t}^{US} - X_{T_t}^{JP}$
  - Consistent and simultaneous modeling of all currency pairs (not limited to 2 economies).


- Reverse engineer the pricing kernel of US, UK, and Japan using currency options on dollar-yen, dollar-pound, and pound-yen.
Limitations and extensions

- Recall the rule of thumb for model design:

\[ \ln{S_t/S_0} = (r - q)t + \sum_{k=1}^{K} \left( b^k X^k - k_x^k(b^k)T^k_t \right), \]

- \( X^k \) are orthogonal. Generate correlation among different factors via factor loadings \( b^k \).
- Limitation: Constant factor loading \( (b^k) \), together with orthogonality assumption, puts restrictions on co-movements across assets.

- Future work: How to allow flexible correlation dynamics (with independent variation)?
  - Each asset (economic shock) has its own business clock.
  - How to model the co-movements of business clocks of multiple assets (economic sources)?
  - If economics ask for it, we may also need to get out of the exponential-affine setup:

Concluding remarks

- Modeling security returns with (time-changed) Lévy processes enjoys three key virtues:
  - **Generality**: Lévy process can be made to capture any return innovation distribution; applying time changes can make this distribution vary stochastic over time.
  - **Explicit economic mapping**: Each Lévy component captures shocks from one economic source. Time changes capture the relative variation of the intensities of these impacts.
  - **Tractability**: Combining any tractable Lévy process (with tractable $\psi(u)$) with any tractable activity rate dynamics (with a tractable Laplace) generates a tractable Fourier transform for the time changed Lévy process. The two specifications are separate.

- It is a nice place to start with for generating security return dynamics that are parsimonious, tractable, economically sensible, and statistically performing well.