Option Properties

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Options Markets

(Hull chapter: 9)
Notation

- $S$: stock price. $F_{t,T}$ — time-$t$ forward price with expiry $T$.
- $K$: strike price.
- Today: either time 0 or time $t$.
- $T$: expiry date (or maturity with $t = 0$).
- $\sigma$: Volatility (annualized standard deviation) of stock return.
- $r$: continuously compounded riskfree rate with maturity $T$ (same as option).
- $D$: present value of discrete dividends paid during option’s life.
- $q$: continuously compounded dividend yield during option’s life (for foreign interest rate for currency options).
<table>
<thead>
<tr>
<th>Variable</th>
<th>$c$</th>
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<tbody>
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<td>$S_t$</td>
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<td>$D/q/r_f$</td>
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In examples given below, I use the following benchmark numbers: $S_t = 100; K = 100; \sigma = 20\%; t = 0; T = 2/12; r = 5\%; q = 3\%$. (unless otherwise specified)
Option variation with spot price

Dependence on spot follows from payoff function (when time to maturity is zero).
Option variation with strike price

Dependence on strike follows from payoff function (when time to maturity is zero).
European option variation with time to maturity

Normally increase, but can decline.
American option value always increases with time, given the option to exercise early.
Option variation with volatility

ATM option value is almost linear in volatility. Options market is mainly a market for volatility.
Increasing interest rate or reducing dividend yield increases the growth rate of the stock price.
American v. European options

- The above graphs are all generated for European options, based a simple option pricing model, which we’ll deal with later.

- An American option is worth at least as much as the corresponding European option: $C \geq c$ and $P \geq p$.
  - The difference is due to the extra option you get from an American option: You can exercise any time before the expiry date.
  - Hence, the difference is also referred to as the early exercise premium.

- Pricing American options is a bit harder — we’ll need a numerical algorithm to deal with the early exercise premium.
Obvious arbitrage opportunities on call options

Suppose that \( c_t = 2, S_t = 20, T - t = 1, K = 17, r = 5\%, q = 0 \). Is there an arbitrage opportunity?

Suppose that \( c_t = 21, S_t = 20, T - t = 1, K = 1, r = 5\%, q = 0 \). Is there an arbitrage opportunity?
Bounds on a call option

- **Lower bound:**
  \[ c_t \geq \max[0, S_t e^{-q(T-t)} - Ke^{-r(T-t)}] = \max[0, e^{-r(T-t)}(F_{t,T} - K)]. \]
  - If the call price is negative, you are paid to own an option.
  - If the call price is lower than the value of the forward at the same strike, \( e^{-r(T-t)}(F_{t,T} - K) \), buy the call and short the forward:
    - Today’s value from the transaction is: \(-c_t + e^{-r(T-t)}(F_{t,T} - K) > 0.\)
    - At maturity, the payoff is \((S_T - K)^+ - (S_T - K) = (K - S_T)^+ > 0.\)
  - Point to remember: **An option is worth more than a forward.**

- **Upper bound:**
  \[ c_t \leq S_t e^{-q(T-t)} = e^{-r(T-t)}F_{t,T}. \]
  - If violated, write the call and long a forward at zero strike.
    - Today’s value from the transaction is: \(c_t - e^{-r(T-t)}F_{t,T} > 0.\)
    - At maturity, the payoff is \(- (S_T - K)^+ + S_T = \min(S_T, K) > 0.\)
  - A forward at zero strike is the same as a call at zero strike.
  - A call with positive strike is worth less than a call with zero strike.
  - More generally, \(c(K_1) \leq c(K_2)\) for all \(K_1 \geq K_2.\) — Can you devise an arbitrage trading when this *monotonicity condition* is violated?

- In the presence of discrete dividends, the bounds are:
  \[ [S_t - D - Ke^{-r(T-t)}, S_t - D]. \]
Suppose that $c_t = 3.2$, $S_t = 20$, $T - t = 1$, $K = 17$, $r = 5\%$, $q = 3\%$. Is there an arbitrage opportunity?

- $S_t e^{-q(T-t)} - Ke^{-r(T-t)} = 3.238$.

Suppose that $c_t = 19.5$, $S_t = 20$, $T - t = 1$, $K = 1$, $r = 5\%$, $q = 3\%$. Is there an arbitrage opportunity?

- $S_t e^{-q(T-t)} = 19.41$.

Suppose that $c_t = 19.5$, $S_t = 20$, $T - t = 1$, $K = 1$, $r = 5\%$, $D = 1$. Is there an arbitrage opportunity?

- $S_t - D = 19$. 
Bounds on a put option

- **Lower bound:**
  \[ p_t \geq \max[0, Ke^{-r(T-t)} - S_te^{-q(T-t)}] = \max[0, e^{-r(T-t)}(K - F_{t,T})]. \]
  - If put price is negative, you are paid to own an option.
  - If put price is lower than the value of the corresponding short forward, buy the put and long the forward:
    - Today’s value from the transaction is: \(-p_t - e^{-r(T-t)}(F_{t,T} - K) > 0.\)
    - At maturity, the payoff is \((K - S_T)^+ + (S_T - K) = (S_T - K)^+ > 0.\)
  - An option to sell is worth more than a forward to sell.
  - With discrete dividends, the lower bound is: \(Ke^{-r(T-t)} - S_t + D.\)

- **Upper bound:** \(p_t \leq Ke^{-r(T-t)}.\)
  - If violated, write the put and save \(Ke^{-r(T-t)}\) in the bank. Spend the rest at water park.
    - Today’s value: water slides.
    - At maturity, the return from the bank is \(K.\) The worst possible obligation from writing the put is \(K\) when the company goes bankruptcy \((S_T = 0).\) So you make even in the worst case, make money otherwise. \(K - (K - S_T)^+ = \min(S_T, K) > 0.\)
  - Monotonicity condition: \(p(K_1) \geq p(K_2)\) for all \(K_1 \geq K_2.\)
Arbitrage opportunities on put options

- Suppose that $p_t = 2, S_t = 20, T - t = 1, K = 23, r = 5\%, q = 3\%$. Is there an arbitrage opportunity?
  
  $Ke^{-r(T-t)} - S_te^{-q(T-t)} = 2.47$.

- Suppose that $p_t = 2, S_t = 20, T - t = 1, K = 23, r = 5\%, D = 1$. Is there an arbitrage opportunity?
  
  $Ke^{-r(T-t)} - S_t + D = 2.88$.

- Suppose the put price at $20$ strike is $2$ and at $21$ strike is $1.9$ at the same maturity. Is there an arbitrage opportunity?
Recall the put-call parity condition: The difference between a call and a put equals the forward.

\[ c_t - p_t = e^{-r(T-t)}(F_{t,T} - K_t) = S_te^{-q(T-t)} - K_te^{-r(T-t)} = S_t - D - K_te^{-r(T-t)} \]

Suppose that \( c_t = 8, S_t = 100, T - t = 1, K = 100, r = 0, q = 0. \) Is there an arbitrage opportunity if

- \( p_t = 7. \)
- \( p_t = 8. \)
- \( p_t = 9. \)
Early exercise of American options

- Usually there is some chance that an American option will be exercised early.
  - Exercise when the “exercise value” \((S_t - K)^+\) for call) is higher than the option value \((C_t)\).
- As a result, the American option is more expensive (valuable) than the European option.
- An exception is an American call on a non-dividend paying stock, which should never be exercised early.
  - \(C_t \geq c_t \geq S_t - Ke^{-r(T-t)} \geq S_t - K = \text{exercise value}\)
  - The call option value for a non-dividend paying stock is always no lower than the exercise value.
Cheat sheet: Summary of important relations/bounds

- **Forward pricing:**
  \[
  F_t = e^{(r-q)(T-t)} S_t = e^{r(T-t)} (S_t - D_t),
  \]

- **European options bounds:**
  \[
  c_t \in [e^{-r(T-t)} (F_{t,T} - K)^+, \quad e^{-r(T-t)} F_{t,T}]
  
  p_t \in [e^{-r(T-t)} (K - F_{t,T})^+, \quad e^{-r(T-t)} K]
  \]

- **Put-call parity for European options:**
  \[
  c_t - p_t = e^{-r(T-t)} (F_{t,T} - K_t) = e^{-q(T-t)} S_t - e^{-r(T-t)} K.
  \]

- **Put-call inequality for American options:**
  \[
  S_t - D - K \leq C_t - P_t \leq S_t - K e^{-r(T-t)}
  
  S_t e^{-q(T-t)} - K \leq C_t - P_t \leq S_t - K e^{-r(T-t)}
  \]

- **Monotonicity for both European and American options:**
  \[
  p(K_1) \geq p(K_2), \quad c(K_1) \leq c(K_2), \quad \text{for all} \quad K_1 > K_2.
  \]

- **Most important:** Know how to trade against violations of these conditions.