The Black-Scholes Model

Liuren Wu

Options Markets
Black and Scholes (1973) and Merton (1973) derive option prices under the following assumption on the stock price dynamics,

\[ dS_t = \mu S_t dt + \sigma S_t dW_t \]  

(explained later)

The binomial model: Discrete states and discrete time (The number of possible stock prices and time steps are both finite).

The BMS model: Continuous states (stock price can be anything between 0 and \( \infty \)) and continuous time (time goes continuously).

Scholes and Merton won Nobel price. Black passed away.

BMS proposed the model for stock option pricing. Later, the model has been extended/twisted to price currency options (Garman&Kohlhagen) and options on futures (Black).

I treat all these variations as the same concept and call them indiscriminately the BMS model (combine chapters 13&14).
Primer on continuous time process

\[ dS_t = \mu S_t dt + \sigma S_t dW_t \]

- The driver of the process is \( W_t \), a Brownian motion, or a Wiener process.
- The process \( W_t \) generates a random variable that is normally distributed with mean 0 and variance \( t \), \( \phi(0, t) \). (Also referred to as Gaussian.)
- The process is made of independent normal increments \( dW_t \sim \phi(0, dt) \).
  - “\( d \)” is the continuous time limit of the discrete time difference (\( \Delta \)).
  - \( \Delta t \) denotes a finite time step (say, 3 months), \( dt \) denotes an extremely thin slice of time (smaller than 1 milisecond).
  - It is so thin that it is often referred to as instantaneous.
  - Similarly, \( dW_t = W_{t+dt} - W_t \) denotes the instantaneous increment (change) of a Brownian motion.
- By extension, increments over non-overlapping time periods are independent: For \( t_1 > t_2 > t_3 \), \( (W_{t_3} - W_{t_2}) \sim \phi(0, t_3 - t_2) \) is independent of \( (W_{t_2} - W_{t_1}) \sim \phi(0, t_2 - t_1) \).
Properties of a normally distributed random variable

\[ dS_t = \mu S_t dt + \sigma S_t dW_t \]

- If \( X \sim \phi(0, 1) \), then \( a + bX \sim \phi(a, b^2) \).
- If \( y \sim \phi(m, V) \), then \( a + by \sim \phi(a + bm, b^2 V) \).
- Since \( dW_t \sim \phi(0, dt) \), the instantaneous price change
  \[ dS_t = \mu S_t dt + \sigma S_t dW_t \sim \phi(\mu S_t dt, \sigma^2 S_t^2 dt) \]
- The instantaneous return \( \frac{dS}{S} = \mu dt + \sigma dW_t \sim \phi(\mu dt, \sigma^2 dt) \).

  - Under the BMS model, \( \mu \) is the annualized mean of the instantaneous return — instantaneous mean return.
  - \( \sigma^2 \) is the annualized variance of the instantaneous return — instantaneous return variance.
  - \( \sigma \) is the annualized standard deviation of the instantaneous return — instantaneous return volatility.
Geometric Brownian motion

\[ \frac{dS_t}{S_t} = \mu dt + \sigma dW_t \]

- The stock price is said to follow a geometric Brownian motion.
- \( \mu \) is often referred to as the drift, and \( \sigma \) the diffusion of the process.
- Instantaneously, the stock price change is normally distributed, \( \phi(\mu S_t dt, \sigma^2 S_t^2 dt) \).
- Over longer horizons, the price change is lognormally distributed.
- The log return (continuous compounded return) is normally distributed over all horizons:
  \[ d \ln S_t = (\mu - \frac{1}{2} \sigma^2) dt + \sigma dW_t \]
    - \( d \ln S_t \sim \phi(\mu dt - \frac{1}{2} \sigma^2 dt, \sigma^2 dt) \).
    - \( \ln S_t \sim \phi(\ln S_0 + \mu t - \frac{1}{2} \sigma^2 t, \sigma^2 t) \).
    - \( \ln S_T / S_t \sim \phi((\mu - \frac{1}{2} \sigma^2)(T - t), \sigma^2(T - t)) \).
- Integral form: \( S_t = S_0 e^{\mu t - \frac{1}{2} \sigma^2 t + \sigma W_t} \), \( \ln S_t = \ln S_0 + \mu t - \frac{1}{2} \sigma^2 t + \sigma W_t \).
Simulate 100 stock price sample paths

\[ dS_t = \mu S_t \, dt + \sigma S_t \, dW_t, \quad \mu = 10\%, \sigma = 20\%, S_0 = 100, t = 1. \]

- Stock with the return process:  \( d \ln S_t = (\mu - \frac{1}{2} \sigma^2) \, dt + \sigma \, dW_t. \)
- Discretize to daily intervals  \( dt \approx \Delta t = 1/252. \)
- Draw standard normal random variables  \( \varepsilon(100 \times 252) \sim \phi(0, 1). \)
- Convert them into daily log returns:  \( R_d = (\mu - \frac{1}{2} \sigma^2) \Delta t + \sigma \sqrt{\Delta t} \varepsilon. \)
- Convert returns into stock price sample paths:  \( S_t = S_0 e^{\sum_{d=1}^{252} R_d}. \)
The key idea behind BMS

- The option price and the stock price depend on the same underlying source of uncertainty.

- The Brownian motion dynamics imply that if we slice the time thin enough ($dt$), it behaves like a binominal tree.

- Reversely, if we cut $\Delta t$ small enough and add enough time steps, the binomial tree converges to the distribution behavior of the geometric Brownian motion.

  - Under this thin slice of time interval, we can combine the option with the stock to form a riskfree portfolio.
  - Recall our hedging argument: Choose $\Delta$ such that $f - \Delta S$ is riskfree.
  - The portfolio is riskless (under this thin slice of time interval) and must earn the riskfree rate.
  - **Magic**: $\mu$ does not matter for this portfolio and hence does not matter for the option valuation. Only $\sigma$ matters.
    - We do not need to worry about risk and risk premium if we can hedge away the risk completely.
The hedging argument mathematically leads to the following partial differential equation:

\[
\frac{\partial f}{\partial t} + (r - q)S \frac{\partial f}{\partial S} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 f}{\partial S^2} = rf
\]

At nowhere do we see \( \mu \). The only free parameter is \( \sigma \) (as in the binominal model).

Solving this PDE, subject to the terminal payoff condition of the derivative (e.g., \( f_T = (S_T - K)^+ \) for a European call option), BMS can derive analytical formulas for call and put option value.

Similar formula had been derived before based on distributional (normal return) argument, but \( \mu \) (risk premium) was still in.

The realization that option valuation does not depend on \( \mu \) is big. Plus, it provides a way to hedge the option position.
The BMS formulae

\[
  c_t = S_t e^{-q(T-t)} N(d_1) - Ke^{-r(T-t)} N(d_2),
  
  p_t = -S_t e^{-q(T-t)} N(-d_1) + Ke^{-r(T-t)} N(-d_2),
\]

where

\[
  d_1 = \frac{\ln(S_t/K) + (r-q)(T-t) + \frac{1}{2} \sigma^2 (T-t)}{\sigma \sqrt{T-t}},
  
  d_2 = \frac{\ln(S_t/K) + (r-q)(T-t) - \frac{1}{2} \sigma^2 (T-t)}{\sigma \sqrt{T-t}} = d_1 - \sigma \sqrt{T-t}.
\]

Black derived a variant of the formula for futures (which I like better):

\[
  c_t = e^{-r(T-t)} [F_t N(d_1) - KN(d_2)],
\]

with \(d_{1,2} = \frac{\ln(F_t/K) \pm \frac{1}{2} \sigma^2 (T-t)}{\sigma \sqrt{T-t}}\).

- Recall: \(F_t = S_t e^{(r-q)(T-t)}\). Use forward price \(F_t\) to accommodate various carrying costs/benefits.

- Once I know call value, I can obtain put value via put-call parity:
  \[
  c_t - p_t = e^{-r(T-t)} [F_t - K_t].
  \]
Cumulative normal distribution

\[ c_t = e^{-r(T-t)} [F_t N(d_1) - KN(d_2)], \quad d_{1,2} = \frac{\ln(F_t/K) \pm \frac{1}{2} \sigma^2(T-t)}{\sigma \sqrt{T-t}} \]

- \( N(x) \) denotes the cumulative normal distribution, which measures the probability that a normally distributed variable with a mean of zero and a standard deviation of 1 (\( \phi(0,1) \)) is less than \( x \).

- See tables at the end of the book for its values.

- Most software packages (including excel) has efficient ways to computing this function.

- Properties of the BMS formula:
  - As \( S_t \) becomes very large or \( K \) becomes very small, \( \ln(F_t/K) \uparrow \infty \), \( N(d_1) = N(d_2) = 1 \). \( c_t = e^{-r(T-t)} [F_t - K] \).
  - Similarly, as \( S_t \) becomes very small or \( K \) becomes very large, \( \ln(F_t/K) \uparrow -\infty \), \( N(-d_1) = N(-d_2) = 1 \). \( p_t = e^{-r(T-t)} [-F_t + K] \).
Options on what?

Why does it matter?

- As long as we assume that the underlying security price follows a geometric Brownian motion, we can use (some versions) of the BMS formula to price European options.

- Dividends, foreign interest rates, and other types of carrying costs may complicate the pricing formula a little bit.

- A simpler approach: Assume that the underlying futures/forwards price (of the same maturity of course) process follows a geometric Brownian motion.

- Then, as long as we observe the forward price (or we can derive the forward price), we do not need to worry about dividends or foreign interest rates — They are all accounted for in the forward pricing.

- Know how to price a forward, and use the Black formula.
Implied volatility

\[ c_t = e^{-r(T-t)} \left[ F_t N(d_1) - K N(d_2) \right], \quad d_{1,2} = \frac{\ln(F_t/K) \pm \frac{1}{2} \sigma^2 (T-t)}{\sigma \sqrt{T-t}} \]

- Since \( F_t \) (or \( S_t \)) is observable from the underlying stock or futures market, \((K, t, T)\) are specified in the contract. The only unknown (and hence free) parameter is \( \sigma \).

- We can estimate \( \sigma \) from time series return. (standard deviation calculation).

- Alternatively, we can choose \( \sigma \) to match the observed option price — implied volatility (IV).

- There is a one-to-one, monotonic correspondence between prices and implied volatilities.
  - As long as the option price does not allow arbitrage against cash, there exists a solution for a positive implied volatility that can match the price.

- Traders and brokers often quote implied volatilities rather than dollar prices.
  - More stable; more informative; excludes arbitrage

- The BMS model says that \( IV = \sigma \). In reality, the implied volatility calculated from different options (across strikes, maturities, dates) are usually different.
Violations of BMS assumptions

- The BMS model says that \( IV = \sigma \). In reality, the implied volatility calculated from different options (across strikes, maturities, dates) are usually different.
- These difference indicates that in reality the security price dynamics differ from the BMS assumptions:
  - **Jumps**: BMS assume that the security price moves by a small amount (diffusion) over a short time interval. In reality, sometimes the market can jump by a large amount in an instant.
    - With jumps, returns are no longer normally distributed, but tend to have fatter tails, and sometimes can be asymmetric (skewed).
    - Implied volatility at different strikes will be different.
  - **Stochastic volatility**: The volatility \( \sigma \) of a security is not constant, but varies randomly over time, and can be correlated with the return move.
    - Implied volatilities will change over time.
    - Stochastic volatility also induces return non-normality.
    - Return-volatility correlation induces return distribution asymmetry.
- Other sources of variations such as credit risk for individual stock and emerging market currency, crash risk for aggregate market index...
Plots of option implied volatilities across different strikes at the same maturity often show a smile or skew pattern, reflecting deviations from the return normality assumption.

A smile implies that the probability of reaching the tails of the distribution is higher than that from a normal distribution. ⇒ Fat tails, or (formally) leptokurtosis.

A negative skew implies that the probability of downward movements is higher than that from a normal distribution. ⇒ Negative skewness in the distribution.
At the-money option implied volatilities vary strongly over time, higher during crises and recessions.
Stochastic skewness on stock indexes and currencies

Implied volatility spread between 80% and 120% strikes

10-delta call minis 10-delta put implied volatility

Return skewness also varies over time.
Second-generation option pricing models strive to add new features to capture the observed implied volatility behaviors.

- **Jumps**: BMS uses Brownian motion to capture continuous price movements, second-generation models use a more general class of processes called *Lévy process* to capture both continuous and discontinuous movements.

- **Stochastic volatility**: MBS assumes constant volatility for the Brownian motion, second-generation models allow the intensity of the Lévy processes to vary stochastically over time.
  
  - Use the concept of *time change* to capture the mapping between calendar clock and business (activity) clock.

- **The doctoral class** provides guidance on how to design models based on observed features and how to price options under newly designed models.
Summary

- Understand the basic properties of normally distributed random variables.
- Map a stochastic process to a random variable.
- Understand the link between BMS and the binomial model.
- Memorize the BMS formula (any version).
- Understand forward pricing and link option pricing to forward pricing.
- Can go back and forth with the put-call parity conditions, lower and upper bounds, either in forward or in spot notation.
- Understand the general implications of the implied volatility plots.