Forward and Futures: Summary and Exercises
Liuren Wu

Forward payoff: If you long a forward on an asset with a delivery price $K$, and the underlying spot price of the asset at expiry (time $T$), then the payoff you have from this investment is $(S_T - K)$. If you short this forward contract, your payoff is $(K - S_T)$.

Examples:

- Suppose you are long 1 million forward contracts on Google with a delivery price of $600 and an expiry date of March 14th, 2007. Plot your payoff as a function of possible Google stock prices on March 14th, 2007 (time $T$). (Today is sometime 2006).

  Answer: The payoff as a function of the stock price at time $T$ is: $(S_T - 600) \times 1$ million.

- Following the above example, suppose the Google stock price on 3/14 is $650, what will be your payoff? If you are short 1 million of this contract instead, what will be your payoff?

  Answer: If $S_T = 650$, the payoff for a long position is $650 - 600 = 50$ million. The payoff of a short (1 million) position is $-50$ million.

Forward position value: If you long a forward on an asset with a delivery price $K$ and expiry date $T$. The current forward price at the same expiry date is $F(t,T)$. The current riskfree interest rate (continuously compounding) over the same maturity is $r(t,T)$. Then, the current (time-$t$) value of your forward position is $e^{-r(t,T)(T-t)} (F(t,T) - K)$. The value of a short forward position is the opposite: $-e^{-r(t,T)(T-t)} (F(t,T) - K)$.

Examples:

- Suppose you short one million forward contracts with a delivery price of $1.30. The current forward price at the same expiry date is $1.60. Assume a continuously compounding riskfree rate of 5%, and a maturity for the forward contract of two years.

  Answer: Value of short 1 million contract with delivery price of $1.30 is (in millions):

  $$-e^{-r(t,T)(T-t)} (F(t,T) - K) = -e^{-0.05 \times 2} (1.60 - 1.30) = -0.27145.$$ 

Forward pricing: The forward price of a contract $F(t,T)$ should be equal to the cost of buying the underlying security and carrying over to the expiry date. The usual cost includes interest cost (as you are buying now
instead at expiry) and storage cost (if it is costly to store). There are also benefits to buy today, such as dividend receipts from buying a stock today, and interest receipts from buying a foreign currency today. If the forward price is higher than the cost of buy and carry, you can short the forward contract as the current (high) price and long the replication portfolio (buy today and carry it over to expiry) to lock in a profit for free. If the forward price is lower than the buy-carry cost, you can long the forward contract at the current (low) forward price and short the replication (short sell the underlying security). Examples:

- Consider a forward contract on the IBM stock. Assume that the stock is currently traded at $100 per share. IBM pays quarterly dividend of $1 per share in the next 5 years (no change). The continuously compounding interest rate is flat at 5% (across all maturities). Compute the forward price of the IBM stock at one-, and two-year maturities.

Answer: For IBM stock, the cost is interest cost, and the benefit is dividends. The forward prices at one- and two-year maturities are, respectively,

\[
F(t, t+1) = 100e^{0.05\times1} - \left( e^{0.05\times3/4} + e^{0.05\times2/4} + e^{0.05\times1/4} + 1 \right) = 101.05.
\]

\[
F(t, t+2) = 100e^{0.05\times2} - \left( e^{0.05\times7/4} + e^{0.05\times6/4} + e^{0.05\times5/4} + e^{0.05\times4/4} \right)
- \left( e^{0.05\times3/4} + e^{0.05\times2/4} + e^{0.05\times1/4} + 1 \right) = 102.16.
\]

- Following the above example, if the market quote for the one-year forward contract is at $103.00 (with zero bid-ask spread), what can you do to lock in a profit?

Answer: Since the market quote (103) is higher than the replication (buy & carry) cost (101.05), we should sell high and buy low to lock in a profit. “Sell high” here means to go short on the forward contract, “buy low” means to buy the stock and carry it over to the expiry date (one year later). Specifically, today we (1) sign the forward contract to take the short position on 1 share of IBM stock, (2) borrow $100 from a bank at an interest rate of 5%, and buy one share of IBM stock. [We should do millions of shares of this, but let’s use one share as an example]. At expiry (one year later), the short forward contract expires, and we need to sell 1 share of IBM for $103 (the delivery price) to satisfy our short-forward obligation. Since we bought one share of IBM one year ago, we can use that one share of IBM to cover the selling position. Furthermore, we need to pay back the $100 loan, which will become $100e^{0.05\times1} = 105.13 where 5.13 is the interest cost for your borrowing. Finally, buying the stock one year ago also generates four $1 dividend payments at each quarter. If we save these payments in the bank at the same 5% rate, you will receive \( 4 \left( e^{0.05\times3/4} + e^{0.05\times2/4} + e^{0.05\times1/4} + 1 \right) = 4.0761 \) in cash at the expiry (4 dividend plus interest). Take everything together, you use your IBM stock to satisfy your forward obligation, hence you do not have any IBM exposure left—whether the IBM stock price is at $400 or $40 does not make a difference to you. Consolidating your cash in- and out-flows leaves you: 103 – 105.13 + 4.0761 = 1.9461. That is what you end up with from all the trading, which is not bad given that it is completely riskfree (it does not depend on how IBM stock performs).

- Consider another case where the market quote for the one-year forward contract is $99. How can you trade to lock in a profit?

Answer: In this case, the forward quote is lower than the replication cost, so we would long the forward contract and short the replication, i.e., short selling the IBM stock. By short selling 1 share of IBM stock, we receive $100, which we save in the bank. At expiry, we pay $99 to receive 1 share of IBM
Consider an opposite example where the two-year forward market quote is $1.89. We return this IBM share to the original lender, and we pay the lender back all the dividends during the past year (plus interests), which amounts to 

$$\left(1e^{r_1 \times 3/4} + 1e^{r_2 \times 2/4} + 1e^{r_3 \times 1/4} + 1\right) = 4.0761.$$ 

That is, by borrowing 1 share of IBM stock one year ago, we need to return the share plus all the dividends we receive (with interests). The money $100 we saved one year ago becomes $100e^{0.05 \times 1} = 105.13$. In total, our cashflow is: $-99 - 4.0761 + 105.13 = 2.0539$. That’s what you are left with after the exercise, and it does not depend on how IBM stock price moves since you do not have any IBM stock exposure left (you bought a share and gave it back to the original lender, and hence have nothing left on IBM).

- Consider a two-year forward contract on the dollar price of pound. The current exchange rate is $1.90 per pound. The continuously compounding US dollar interest rate is flat at 4%, and the corresponding pound rate is at 5%. What should be the forward price at the two-year maturity?

  Answer: $F(t, T) = S_t e^{(r_d - r_f)(T-t)} = 1.90 \times e^{(0.04 - 0.05) \times 2} = 1.8624$. Here, buying pound costs dollar. Hence, dollar rate is the cost, pound rate is the benefit.

- Following the above example, if the market quote for the two-year forward is $1.89, what can you do to lock in a profit?

  Answer: The forward price is higher (1.89) than the replication cost (1.86). Hence, I would short the forward today and buy the pound. Suppose I short the forward for one pound, I would need to buy $1e^{-0.05 \times 2} = 0.90484$ pound today and save this pound in the bank to earn a 5% pound interest rate. $0.90484$ grows at 5% for two years becomes $1: 0.90484e^{0.05 \times 2} = 1.0$. To buy $0.90484$ pound, I need to spend $0.90484 \times 1.90 = 1.7192$ dollars as each pound is worth $1.90 dollar now. I would borrow this money ($1.7192$) from the bank and pay dollar interest rate (4%) on this debt.

  At expiry, my pound savings grow to one pound, with which I can satisfy my short forward obligation and receive the delivery price of $1.89. My borrowed dollar (debt) $1.7192$ grows to $1.7192e^{0.04 \times 2} = 1.8624$ based on the 4% dollar rate. My net cashflow is: $1.89 - 1.8624 = 0.0276$, which is a small profit that I lock in with no currency exposure.

  The key in this example is that we do not need to buy one pound to have one pound at expiry because pound can grow over time with its interest rate.

- Consider an opposite example where the two-year forward market quote is $1.84. What can you do to lock in a profit?

  Answer: Since the market quote (1.84) is lower than the replication cost (1.86), I would go long on the forward and short sell the pound. Long the forward implies that I would receive one pound at expiry. Hence, I can borrow some pound today knowing that I would have 1 pound to pay back at expiry. But I cannot borrow one pound as the debt will grow. I can only borrow $1e^{-0.05 \times 2} = 0.90484$ pound as this debt will grow to 1 pound at expiry with the 5% rate on pound. With the borrowed pound, I sell it to the spot market to make $1.90 per pound and hence $0.90484 \times 1.90 = 1.7192$ dollars total. I save these dollars in the bank and make 4% dollar rate on them. At expiry, the dollar saving grows to $1.7192e^{0.04 \times 2} = 1.8624$. My pound debt grows to one pound. To satisfy the long forward obligation, I spend $1.84 (the delivery price) to buy one pound and use this pound to pay back my pound debt. What is left is $1.8624$ I receive for my dollar savings and $1.84$ I pay for the long forward delivery. My net profit is $1.8624 - 1.84 = 0.0224$, a small but nevertheless sure profit regardless of how pound or dollar moves.