Assume zero interest rates \((r)\) and zero dividends \((q)\) wherever applicable. All options are European.

1. Consider stock with a current price \((S_t)\) of $50 and a constant annualized return volatility \((\sigma)\) of 40%. The stock price evolves according to the geometric Brownian motion.

   \[ \frac{\ln S_T}{S_t} = (r - q - \frac{1}{2}\sigma^2)(T - t) + \sigma(W_T - W_t). \]

   (a) (10) Based on the above assumption, we can write the risk-neutral log stock return dynamics as
   \[ \ln S_T/S_t = (r - q - \frac{1}{2}\sigma^2)(T - t) + \sigma(W_T - W_t). \]
   The return is normally distributed, what are mean and variance of the return over a time horizon of three months \((T - t = 3/12)\)? [in concrete numbers]

   (b) (10) Using the approach discussed in class, construct a two-step binomial tree to approximate the stock price dynamics, with each step being 3 months. List the stock price at each node at three and six months.

   (c) (10) Compute the risk-neutral probability of going up an going down at each step. [The probability of going up is the same at each node. So just calculate once.]

   (d) (10) Based on the binomial tree, compute (i) the current value and (ii) the delta of a call option on the stock with a maturity of three months and a strike price of $55. [Just one step]

   (e) According to the Black-Scholes formula, the delta of the call option at $55 strike and three-month maturity is 0.3532. I have also computed \(N(d_2) = 0.2821\). Compute:
   i. (10) The Black-Scholes (i) value and (ii) vega of the call option.
   ii. (15) The Black-Scholes (i) value, (ii) delta, and (iii) vega of a put with the same strike and maturity.
   iii. (5) Compared to the 3-month $55 strike call, should the following call options have higher or lower delta? (i) a call at $55 strike and 1-year maturity. (ii) a call at $60 strike and 3-month maturity.

2. You have a portfolio of options on the same stock, with a delta 10 million and vega $-400 million.

   (a) (5) If the stock price suddenly falls by one dollar while the volatility does not change, how much do you expect your portfolio value to change?

   (b) (5) If the stock price does not change but the volatility suddenly goes up by one percentage point (0.01, or 1%), how much do you expect your portfolio value to change?

   (c) (10) If you want to alter your risk exposure using (i) the underlying stock and (ii) a put option with a delta of $-0.5$ and a vega of 10. How many of these two contracts do you need to balance your portfolio to delta and vega neutral?

3. (10) Comparing to a normal distribution benchmark, investors expect that the stock has a higher (risk-adjusted) probability of generating large positive returns and a lower probability of generating large negative returns. Plot the implied volatility \((IV)\) at a fixed maturity as a function of the strike \((K)\) that reflects this view of the investors. [A schematic plot, no numbers needed.]