

P&L Attribution and Risk Management

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Options Markets

Outline

- 1 P&L attribution via the BSM model
- 2 Delta
- 3 Vega
- 4 Gamma
- 5 Static hedging

- If we own a portfolio of European options at the same maturity, we know how to construct the payoff function of the portfolio at expiry.
- Before the option expires, the option prices vary as the underlying price changes and as volatility changes.
- For risk management, it is important to know as the underlying price goes up or down by 1%, or as the underlying return volatility goes up or down by 1%, how much the portfolio value will change.
- If the portfolio value can vary a lot (the portfolio is very risky, volatile), risk managers must propose ways to reduce the risk, either by reducing/unloading positions, or by *hedging*.

P&L attribution via the BSM model

- The common practice is to analyze and manage the options risk via the BSM pricing relation, $B(t, S_t, I_t)$.
 - B denotes the BSM pricing formula (for an option at K, T)).
 - The option value vary over time due to variations in calendar time (t), underlying security price (S_t), the implied volatility of the option (I_t).
 - How calendar time moves forward is known; but the variation of S_t and I_t in the future is unknown and must be managed.
- One can perform a *Taylor expansion* of the option value change over a short time interval (say one day):

$$\begin{aligned}\frac{\Delta B_t}{\Delta t} &= \frac{\partial B_t}{\partial t} \Delta t + \frac{\partial B_t}{\partial S_t} \Delta S_t + \frac{\partial B_t}{\partial I_t} \Delta I_t \\ &\quad + \frac{1}{2} \frac{\partial^2 B}{\partial S_t^2} (\Delta S_t)^2 + \frac{\partial^2 B}{\partial S_t \partial I_t} (\Delta S_t)(\Delta I_t) + \frac{1}{2} \frac{\partial^2 B}{\partial I_t^2} (\Delta I_t)^2\end{aligned}$$

- The partial derivatives capture (risk) exposures to time decay ($\frac{\partial B_t}{\partial t}$, *theta*), price movement ($\frac{\partial B_t}{\partial S_t}$, *delta*), volatility movement ($\frac{\partial B_t}{\partial I_t}$, *vega*), second-order effects ($\frac{\partial^2 B}{\partial S_t^2}$, *gamma*; $\frac{\partial^2 B}{\partial S_t \partial I_t}$, *vanna*; $\frac{\partial^2 B}{\partial I_t^2}$, *volga*)
- They are often referred to as *option greeks*.

Risk management via BSM greeks

$$\frac{\Delta B_t}{\Delta t} = \frac{\partial B_t}{\partial t} \Delta t + \frac{\partial B_t}{\partial S_t} \Delta S_t + \frac{\partial B_t}{\partial I_t} \Delta I_t + \frac{1}{2} \frac{\partial^2 B}{\partial S_t^2} (\Delta S_t)^2 + \frac{\partial^2 B}{\partial S_t \partial I_t} (\Delta S_t)(\Delta I_t) + \frac{1}{2} \frac{\partial^2 B}{\partial I_t^2} (\Delta I_t)^2$$

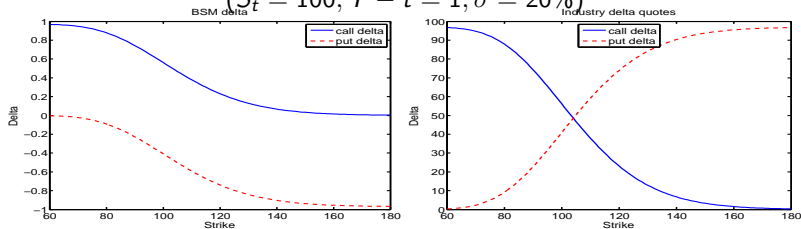
- If we can estimate all the greeks (risk exposures) of an option (portfolio), we would know how much the portfolio value can change if some risk changes by a certain amount.
- If we form a portfolio that cancels out all risk exposures, the portfolio value will not vary much no matter what varies — This is a very safe portfolio.
- If we have a stock option portfolio with a delta of 1bn, it means that the portfolio can lose by \$1bn dollars if the stock price goes down by \$1.
 - The risk manager can remove this risk by selling 1bn share of the stock.
- If the portfolio has a delta exposure of -1bn, it means that the portfolio can lose by \$1bn dollars if the security price goes up by \$1.
- If the portfolio has a vega exposure of -1bn, the portfolio can lose \$10million if the volatility goes up by 0.01 (or one percentage point).

The BSM Delta

The BSM delta of European options (*Can you derive them?*):

$$\Delta_c \equiv \frac{\partial c_t}{\partial S_t} = e^{-q\tau} N(d_1), \quad \Delta_p \equiv \frac{\partial p_t}{\partial S_t} = -e^{-q\tau} N(-d_1)$$

$$(S_t = 100, T - t = 1, \sigma = 20\%)$$



- Industry quotes the delta in absolute percentage terms (right panel).
- *Which of the following is out-of-the-money?* (i) 25-delta call, (ii) 25-delta put, (iii) 75-delta call, (iv) 75-delta put.
- The strike of a 25-delta call is close to the strike of: (i) 25-delta put, (ii) 50-delta put, (iii) 75-delta put.

Delta as a moneyness measure

Different ways of measuring moneyness:

- K (relative to S or F): Raw measure, not comparable across different stocks.
- K/F : better scaling than $K - F$.
- $\ln K/F$: more symmetric under BSM.
- $\frac{\ln K/F}{\sigma\sqrt{(T-t)}}$: standardized by volatility and option maturity, comparable across stocks. Need to decide what σ to use (ATMV, IV, 1).
- d_1 : a standardized variable.
- d_2 : Under BSM, this variable is the truly standardized normal variable with $\phi(0,1)$ under the risk-neutral measure.
- **delta**: Used frequently in the industry, quoted in absolute percentages.
 - Measures moneyness: Approximately the percentage chance the option will be in the money at expiry.
 - Reveals your underlying exposure (how many shares needed to achieve delta-neutral).

Delta hedging

- Example: A bank has sold for \$300,000 a European call option on 100,000 shares of a nondividend paying stock, with the following information: $S_t = 49$, $K = 50$, $r = 5\%$, $\sigma = 20\%$, $(T - t) = 20\text{weeks}$, $\mu = 13\%$.
 - What's the BSM value for the option? $\rightarrow \$2.4$
 - What's the BSM delta for the option? $\rightarrow 0.5216$.
- **Delta hedging**: Buy 52,000 share of the underlying stock now. Adjust the shares over time to maintain a **delta-neutral portfolio**.

Delta hedging with futures

- The delta of a futures contract is $e^{(r-q)(T-t)}$.
- The delta of the option with respect to (wrt) futures is the delta of the option over the delta of the futures.
- The delta of the option wrt futures (of the same maturity) is

$$\begin{aligned}\Delta_{c/F} &\equiv \frac{\partial c_t}{\partial F_{t,T}} = \frac{\partial c_t / \partial S_t}{\partial F_{t,T} / \partial S_t} = e^{-r\tau} N(d_1), \\ \Delta_{p/F} &\equiv \frac{\partial p_t}{\partial F_{t,T}} = \frac{\partial p_t / \partial S_t}{\partial F_{t,T} / \partial S_t} = -e^{-r\tau} N(-d_1).\end{aligned}$$

- Whenever available (such as on indexes, commodities), using futures to delta hedge can potentially reduce transaction costs.

OTC quoting and trading conventions for currency options

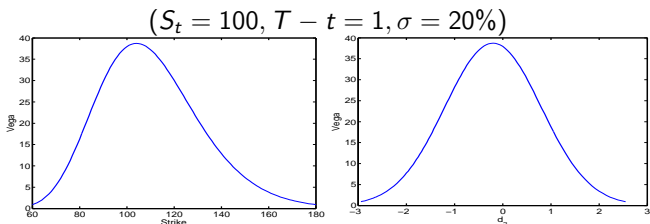
- Options are quoted at fixed time-to-maturity (not fixed expiry date).
- Options at each maturity are not quoted in invoice prices (dollars), but in the following format:
 - **Delta-neutral straddle implied volatility (ATMV):**
A straddle is a portfolio of a call & a put at the same strike. The strike here is set to make the portfolio delta-neutral $\Rightarrow d_1 = 0$.
 - **25-delta risk reversal:** $RR_{25} = IV(\Delta_c = 25) - IV(\Delta_p = 25)$.
 - **25-delta butterfly spreads:**
 $BF_{25} = (IV(\Delta_c = 25) + IV(\Delta_p = 25))/2 - ATMV$.
 - Risk reversals and butterfly spreads at other deltas, e.g., 10-delta.
- When trading, invoice prices and strikes are calculated based on the BSM formula.
- The two parties exchange both the option and the underlying delta.
 - The trades are delta-neutral.

The BSM vega

- Vega (ν) is the rate of change of the value of a derivatives portfolio with respect to volatility — it is a measure of the **volatility exposure**.
- BSM vega: the same for call and put options of the same maturity

$$\nu \equiv \frac{\partial C_t}{\partial \sigma} = \frac{\partial P_t}{\partial \sigma} = S_t e^{-q(T-t)} \sqrt{T-t} n(d_1)$$

$n(d_1)$ is the standard normal probability density: $n(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}$.



Volatility exposure (vega) is higher for at-the-money options.

Vega hedging

- Delta can be changed by taking a position in the underlying.
- To adjust the volatility exposure (vega), it is necessary to take a position in an option or other derivatives.
- Hedging in practice:
 - Traders usually ensure that their portfolios are delta-neutral at least once a day.
 - Whenever the opportunity arises, they improve/manage their vega exposure — options trading is more expensive.
 - As portfolio becomes larger, hedging becomes less expensive.
- Under the assumption of BSM, vega hedging is not necessary: σ does not change. But in reality, it does.
 - Vega hedge is outside the BSM model.

Example: Delta and vega hedging

Consider an option portfolio that is delta-neutral but with a vega of $-8,000$. We plan to make the portfolio both delta and vega neutral using two instruments:

- The underlying stock
- A traded option with delta 0.6 and vega 2.0.

How many shares of the underlying stock and the traded option contracts do we need?

- To achieve vega neutral, we need **long 8000/2=4,000** contracts of the traded option.
- With the traded option added to the portfolio, the delta of the portfolio increases from 0 to $0.6 \times 4,000 = 2,400$.
- We hence also need to **short 2,400** shares of the underlying stock \Rightarrow each share of the stock has a delta of one.

Another example: Delta and vega hedging

Consider an option portfolio with a delta of 2,000 and vega of 60,000. We plan to make the portfolio both delta and vega neutral using:

- The underlying stock
- A traded option with delta 0.5 and vega 10.

How many shares of the underlying stock and the traded option contracts do we need?

- As before, it is easier to take care of the vega first and then worry about the delta using stocks.
- To achieve vega neutral, we need $\text{short/write } 60000/10 = 6000$ contracts of the traded option.
- With the traded option position added to the portfolio, the delta of the portfolio becomes $2000 - 0.5 \times 6000 = -1000$.
- We hence also need to $\text{long } 1000$ shares of the underlying stock.

A more formal setup

Let $(\Delta_p, \Delta_1, \Delta_2)$ denote the delta of the existing portfolio and the two hedging instruments. Let (ν_p, ν_1, ν_2) denote their vega. Let (n_1, n_2) denote the shares of the two instruments needed to achieve the target delta and vega exposure (Δ_T, ν_T) . We have

$$\begin{aligned}\Delta_T &= \Delta_p + n_1\Delta_1 + n_2\Delta_2 \\ \nu_T &= \nu_p + n_1\nu_1 + n_2\nu_2\end{aligned}$$

We can solve the two unknowns (n_1, n_2) from the two equations.

- Example 1: The stock has delta of 1 and zero vega.

$$\begin{aligned}0 &= 0 + n_1 0.6 + n_2 \\ 0 &= -8000 + n_1 2 + 0\end{aligned}$$

$$n_1 = 4000, n_2 = -0.6 \times 4000 = -2400.$$

- Example 2: The stock has delta of 1 and zero vega.

$$\begin{aligned}0 &= 2000 + n_1 0.5 + n_2 \\ 0 &= 60000 + n_1 10 + 0\end{aligned}$$

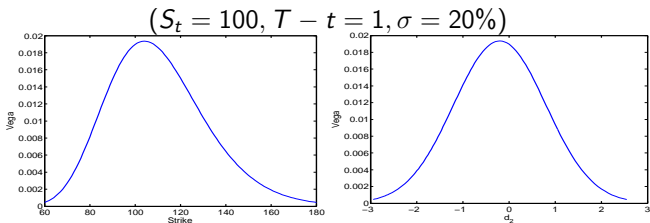
$$n_1 = -6000, n_2 = 1000.$$

- *When do you want to have non-zero target exposures?*

BSM gamma

- Gamma (Γ) is the rate of change of delta (Δ) with respect to the price of the underlying asset.
- The BSM gamma is the same for calls and puts:

$$\Gamma \equiv \frac{\partial^2 c_t}{\partial S_t^2} = \frac{\partial \Delta_t}{\partial S_t} = \frac{e^{-q(T-t)} n(d_1)}{S_t \sigma \sqrt{T-t}}$$



Gamma is high for near-the-money options.

High gamma implies high variation in delta, and hence more frequent rebalancing to maintain low delta exposure.

Gamma hedging

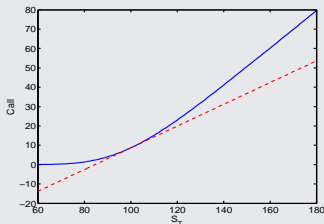
- High gamma implies high variation in delta, and hence more frequent rebalancing to maintain low delta exposure.
- Delta hedging is based on small moves during a very short time period.
 - assuming that the relation between option and the stock is linear locally.
- When gamma is high,
 - The relation is more curved (convex) than linear,
 - The P&L (hedging error) is more likely to be large in the presence of large moves.
- The gamma of a stock is zero.
- We can use traded options to adjust the gamma of a portfolio, similar to what we have done to vega.
- But if we are really concerned about large moves, we may want to try something else.

Vanna and volga risk

- The BMS model has only one risk source: Delta
 - Delta hedge removes all risk.
- When return volatility can also vary stochastically over time, we need to consider both how much it varies and how it co-moves with the security price movement
 - Vanna captures the exposure to the covariation, while volga captures the exposure of the variation of the volatility
- They are the focus of a new generation of option pricing and risk management models
 - At one option maturity, the theta variation can be explained (nearly fully) by volga and vanna variations.
 - The value of straddle, strangle, and risk reversal can be mapped to vega, volga, and vanna exposures.
- Reference: *Option profit and loss attribution and pricing: A new framework*

Dynamic hedging with greeks

The idea of delta and vega hedging is based on a locally linear approximation (partial derivative) of the relation between the derivative portfolio value and the underlying stock price and volatility.



Since the relation is not linear, the hedging ratios change as the environment change.

- I call these types of hedging based on partial derivatives as *dynamic hedging*, which often asks for frequent rebalancing.
- Dynamic hedging works well if
 - The overall relation is close to linear. Hence, the hedging ratio is stable (does not change much) over time.
 - The underlying variable (stock price, volatility) varies smoothly and only changes a little within a certain time interval.

Dynamic versus static hedging

- Dynamic hedging can generate large hedging errors when the underlying variable (stock price) can jump randomly.
 - A large move size per se is not an issue, as long as we know how much it moves — a binomial tree can be very large moves, but delta hedge works perfectly.
 - As long as we know the magnitude, hedging is relatively easy.
 - The key problem comes from **large moves of random size**.
- An alternative is to devise **static hedging** strategies: The position of the hedging instruments does not vary over time.
 - Conceptually not as easy. Different derivative products ask for different static strategies.
 - It involves more option positions. Cost per transaction is high.
 - Monitoring cost is low. Fewer transactions.