



A comprehensive analysis of the short-term interest-rate dynamics

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Received 31 August 2004; accepted 29 March 2005

Available online 4 June 2005

Abstract

This paper provides a comprehensive analysis of the short-term interest-rate dynamics based on three different data sets and two flexible parametric specifications. The significance of nonlinearity in the short-rate drift declines with increasing maturity for the interest-rate series used in the study. Using a flexible diffusion specification and incorporating GARCH volatility and non-normal innovation reduce the need for a nonlinear drift specification. Finally, the nonlinear drift specification performs better than the linear drift specification only when the short-term interest-rate levels reach historical highs.

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JEL classification: C13; C22; G12

Keywords: Short-term interest rates; Nonlinearity; Jumps; GARCH; Stochastic volatility

1. Introduction

The short-term interest rate is a fundamental variable in both theoretical and empirical finance because of its central role in asset pricing. An enormous amount

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of work has been directed towards the understanding of the stochastic behavior of short-term interest rates. Nevertheless, based on different data sets and/or different parametric or nonparametric specifications, these studies have generated confusing and sometimes conflicting conclusions. Fundamental questions remain unanswered: (i) Is the short-rate drift linear or nonlinear? (ii) How sensitive is the conclusion to the choice of interest-rate series and parametric specifications?

In this paper, we address the two questions through a comprehensive analysis based on three different interest-rate series and two flexible parametric specifications. Most empirical studies on the short-rate dynamics use one of the three interest-rate series, or their shorter sample. We investigate how using different interest-rate series affects our conclusions on the interest-rate dynamics. The two parametric specifications encompass most of the short-rate models in the literature. We estimate various constrained versions of the two specifications and study how the conclusions are driven by these parametric constraints.

Within a one-factor diffusion framework, we find that the significance of nonlinearity in the short-rate drift relies crucially on both the specification of the diffusion function and on the choice of data sets. For the same data set, the significance of nonlinearity in the drift function declines as the conditional variance changes from an affine function of the short rate, to a constant-elasticity-volatility (CEV) function, and then to a combined, general form. For the same conditional variance specification, the significance of nonlinearity in the drift function declines as the maturity of the interest-rate series increases from the overnight federal funds rate, to the seven-day Eurodollar deposit rate, and to the three-month Treasury bill yield. Under the general conditional variance specification, we find strong and statistically significant nonlinearity and mean-reversion in the drift function of the federal funds rate. The nonlinearity in the drift function of the seven-day Eurodollar rate is also statistically significant, but not as strong as in the federal funds rate. For the three-month Treasury yield, we cannot reject the linear drift specification against the general nonlinear specification under a general conditional variance function. Only under an affine or CEV variance function do we find statistically significant nonlinearity in the drift function of the three-month Treasury bill yield.

Our second parametric framework incorporates both GARCH volatility and non-normal interest-rate innovation, which we assume follows the generalized error distribution (GED). The GARCH specification captures time-varying volatility, and the GED distribution assumption accounts for potentially discontinuous price movements (jumps). When we estimate the interest-rate dynamics within this more flexible framework, the significance of nonlinearity in the drift function declines further. For both the Treasury yield and the seven-day Eurodollar rate, nonlinearities in the drift function are no longer statistically significant. Only the drift function for the overnight federal funds rate remains strongly nonlinear.

We analyze the source of nonlinearity by comparing the daily likelihood estimates of alternative models. For all three interest-rate series, the superiority of the nonlinear drift over the linear drift specification becomes the most pronounced when the short-term interest-rate level reaches historical highs during the high-inflation period of the early 1980s. The market expects, and experiences, a higher-than-normal speed

of mean reversion under those extreme circumstances, generating nonlinearity in the short-rate dynamics.

With an upward-sloping mean term structure, the three-month Treasury yield is on average higher than the overnight federal funds rate. But during the early 1980s, the yield curve reversed its shape. The three-month Treasury yield was lower than the overnight federal funds rate, showing that the three-month rate had already incorporated part of the market expectation on rapid mean reversion in the short rate process. Such incorporation of the market expectation dampens the movement on the longer-term interest rates and makes the nonlinearity in these longer-term rates less significant.

This paper is organized as follows. Section 2 provides the literature review. Section 3 presents the two parametric frameworks for the interest-rate dynamics. Section 4 describes the relevant features of the interest-rate data and explains the estimation methodology. Section 5 compares the relative performance of the models under different parametric constraints for different interest-rate series. Section 6 estimates a class of two-factor diffusion models with stochastic volatility as a robustness check. Section 7 concludes.

2. Literature review

Numerous studies have investigated the behavior of the short-term interest-rate dynamics, but the conclusions from these studies are far from consensual. Using a semi-nonparametric approach, [Aït-Sahalia \(1996a,b\)](#) finds strong nonlinearity in the drift of daily seven-day Eurodollar deposit rates. [Stanton \(1997\)](#) and [Jiang \(1998\)](#) apply a fully nonparametric estimator on the three-month Treasury bills and find similar nonlinearities in the drift function. The nonparametric estimation in [Conley et al. \(1997\)](#) also identifies nonlinearity in the drift of federal funds rates.

[Pritsker \(1998\)](#) studies the finite-sample properties of Aït-Sahalia's nonparametric test. After adjusting for the high persistence of interest rates, Pritsker finds that the nonlinearity in the drift function is no longer statistically significant. [Chapman and Pearson \(2000\)](#) use simulation analysis to show that even when the true data generating process has a linear drift, the nonparametric estimation methodologies used in the above studies may still generate nonlinear estimates for the drift function. To improve the finite-sample performance of nonparametric methods, [Hong and Li \(2005\)](#) introduce a data transformation method and correct the boundary bias of the kernel estimators. Their test strongly rejects a variety of one-factor diffusion models for the seven-day Eurodollar deposit rate, but they also find that incorporating nonlinearity in the short-rate drift does not improve the goodness of fit.

[Jones \(2003\)](#) reevaluates the empirical evidence for nonlinear drift in the seven-day Eurodollar rate using a Bayesian method. He concludes that the finding of large negative drift for high interest rates depends largely on the chosen prior distributions. Jones identifies nonlinearity only when he uses a flat prior distribution and imposes the stationary condition on the interest-rate dynamics. When he implements an approximate Jeffreys prior, Jones finds virtually no evidence of mean reversion unless stationarity is imposed.

Durham (2003) applies maximum likelihood estimation to the three-month Treasury bill yield and finds that the significance of nonlinearity in the drift function depends on the specification of the diffusion function. Durham identifies a significant nonlinear drift when the diffusion follows the CEV form. When he allows for stochastic volatility, the evidence for nonlinearity or even linear mean reversion disappears in the three-month Treasury yield. Durham concludes that the conditional mean of interest-rate changes is constant but the interest-rate volatility is stochastic.

Many other studies have found that multiple factors govern the dynamics and term structure of interest rates. Examples include Longstaff and Schwartz (1992), Brenner et al. (1996), Koedijk et al. (1997), Andersen and Lund (1997), Ball and Torous (1999), Bali (2000), Christiansen (2005), and Hong et al. (2004). These studies often find that an additional stochastic volatility factor or a GARCH-type process is useful to accommodate the strong conditional heteroscedasticity in short-term interest rates. Hong and Li (2005) find that introducing nonlinearity in the short rate dynamics does not significantly improve the empirical performance of single-factor models, and the main reason for the rejection of one-factor diffusions is the violation of the Markov assumption.

Another stream of related studies, e.g., Das (2001), Johannes (2004), and Piazzesi (2005), identify a jump component in the interest-rate movement. The jump component generates interest-rate innovations that are not normally distributed.

Debates also exist on the choice of data sets for proxying the instantaneous interest rate. Some authors argue against the use of high-frequency data on federal funds rates and seven-day Eurodollar rates. Hamilton (1996) asserts that the federal funds rate displays spurious microstructure effects driven by the Fed policy. Jones (2003) argues that the seven-day Eurodollar deposit rate is not a good proxy for the true short rate because the interest-rate series appear to be very noisy, and contain a transitory element that is not reflected in the volatility of longer-maturity Eurodollar rates. Using longer-maturity interest-rate series is not without criticism, either. Chapman et al. (1999) argue that the three-month Treasury yield is a poor substitute for the instantaneous interest rate when the model under consideration is nonlinear. In this paper, we do not attempt to identify the optimal proxy for the instantaneous interest rate. Instead, we empirically investigate how our conclusions on interest-rate dynamics vary when we use different interest-rate series in our empirical analysis.

3. Interest-rate models

One of the possible sources for the conflicting evidence on the short-rate dynamics is the different parametric or nonparametric specifications underlying the tests in different studies. In this paper, we propose two flexible modeling frameworks that encompass most of the interest-rate specifications used in the literature. The first framework encompasses most one-factor diffusion specifications in the literature. The second framework adopts a discrete-time setting that accommodates both GARCH-type volatility and non-normal interest-rate innovation.

3.1. A one-factor diffusion model

Under the one-factor diffusion framework, we can characterize the interest-rate dynamics by the following stochastic differential equation:

$$dr_t = \mu(r_t)dt + \sqrt{v(r_t)}dW_t, \tag{1}$$

where r_t denotes the interest rate at time t and W_t denotes a standard Brownian motion. The term $\mu(r_t)$ and $v(r_t)$ denote generic instantaneous drift and variance functions, respectively.

Some studies estimate the drift and/or variance functions nonparametrically. However, the nonparametric estimators have been found to suffer from low powers in identifying nonlinearity in the interest-rate dynamics. In this paper, we propose parametric but flexible functional forms for both the drift and variance functions,

$$\mu(r_t) = \alpha_0 + \alpha_1 r_t + \alpha_2 r_t^2 + \alpha_3 r_t^3 + \alpha_4 r_t^4 + \alpha_5 r_t^5 + \alpha_6 r_t^{-1}, \tag{2}$$

$$v(r_t) = \beta_0 + \beta_1 r_t + \beta_2 r_t^{\beta_3}. \tag{3}$$

We can think of the drift function as a Laurent series expansion of a generic function with positive order of five and negative order of one. The variance function combines an affine specification with a CEV specification. Our general specification encompasses most one-factor diffusion models considered in the literature. Table 1 summarizes the studies that are nested in our general specification.

3.2. GARCH volatility and non-normal innovation

Under a discrete-time setting, we supplement the one-factor diffusion specification in Eq. (1) with a model that accommodates both GARCH-type volatility and non-normal innovation. GARCH-volatility captures the well-documented evidence on conditional heteroscedasticity and volatility clustering. Non-normal interest-rate innovation can arise due to discontinuous interest-rate movements.

Our specification takes the following general form:

$$r_t - r_{t-1} = \mu(r_{t-1}) + \varepsilon_t, \tag{4}$$

where $\mu(r_{t-1})$ denotes the conditional mean of interest-rate changes and ε_t denotes the innovation or shocks. We normalize the discrete-time interval to a unit of one.

Table 1
Nested cases of the one-factor diffusion model

Model	Constraints on $\mu(r_t)$	Constraints on $v(r_t)$
Duffie and Kan (1996)	$\alpha_2 = \alpha_3 = \alpha_4 = \alpha_5 = \alpha_6 = 0$	$\beta_2 = 0$
Ahn and Gao (1999)	$\alpha_0 = \alpha_3 = \alpha_4 = \alpha_5 = \alpha_6 = 0$	$\beta_0 = \beta_1 = 0, \beta_3 = 3$
Spencer (1999)	$\alpha_0 = \alpha_1 = \alpha_4 = \alpha_5 = \alpha_6 = 0$	$\beta_0 = \beta_1 = 0, \beta_3 = 3$
Chan et al. (1992)	$\alpha_2 = \alpha_3 = \alpha_4 = \alpha_5 = \alpha_6 = 0$	$\beta_0 = \beta_1 = 0$
Aït-Sahalia, 1996a,b	$\alpha_3 = \alpha_4 = \alpha_5 = 0$	
Jones (2003)	$\alpha_3 = \alpha_4 = \alpha_5 = 0$	$\beta_0 = \beta_1 = 0$

We use the same flexible functional form as in Eq. (2) for the conditional mean $\mu(r_{t-1})$, but incorporate GARCH-type effects to the conditional variance of interest-rate changes, $v_t \equiv E_{t-1}[\varepsilon_t^2]$. We consider two specifications for the conditional variance. The first specification is

$$v_t = \beta_0 + \beta_1 \varepsilon_{t-1}^2 + \beta_2 v_{t-1} + \beta_3 r_{t-1}, \quad (5)$$

where we augment a GARCH(1,1) specification with an affine term on the lagged short-rate level. We label this specification as the GARCH-affine model.

In the second specification, the GARCH effect and the interest-rate-level dependence interact multiplicatively,

$$v_t = h_t r_{t-1}^{\beta_3} \quad \text{with} \quad h_t = \beta_0 + \beta_1 \varepsilon_{t-1}^2 + \beta_2 h_{t-1}. \quad (6)$$

We label this specification as the GARCH-CEV model, given the CEV-type dependence on the interest-rate level.

Apart from the notational differences between the continuous-time and discrete-time models, the specification in Eq. (4) differs from the one-factor diffusion specification in Eq. (1) by allowing GARCH effects in the conditional variance of interest-rate changes. Furthermore, the one-factor diffusion implies normal innovation, but we use a more flexible generalized error distribution (GED) to describe the standardized innovation in Eq. (4), the density function of which is given by

$$f(x) = \frac{\theta \exp\left[-\frac{1}{2} \left|\frac{x}{\lambda}\right|^\theta\right]}{\lambda 2^{(\theta+1)/\theta} \Gamma(1/\theta)}, \quad \lambda = \sqrt{\frac{2^{-2/\theta} \Gamma(1/\theta)}{\Gamma(3/\theta)}}, \quad (7)$$

with $x = \varepsilon_t / \sqrt{v_t}$ denoting the standardized innovation and $\Gamma(\cdot)$ denoting the gamma function. The GED distribution dates back to [Subbotin \(1923\)](#) and has been used in econometric works by [Box and Tiao \(1962\)](#) and [Nelson \(1991\)](#). It nests the normal distribution as a special case with $\theta = 2$. When $\theta < 2$, the distribution has thicker tails than the normal distribution.

4. Data and estimation

Another potential source for the conflicting evidence on the short-rate dynamics is the choice of different interest-rate data series. The literature has mainly used three data sets: the federal funds rate, the seven-day Eurodollar deposit rate, and the three-month Treasury bill yield. To understand how the data choice affects the conclusions on the short-rate dynamics, we estimate the interest-rate models using all three series.

We obtain daily data on all three series. The federal funds rate data start on July 1, 1954 and end on June 30, 1999, for a total of 11,319 daily observations spanning 45 years. The seven-day Eurodollar deposit rates are from June 1, 1973 to February 25, 1995, yielding a total of 5505 daily observations. The three-month Treasury yields cover the period from January 8, 1954 to December 31, 1999, totalling 11,488 daily observations. We download the federal funds rate and the three-month Treasury yield from the Federal Reserve H.15 database. [Yacine Ait-Sahalia](#) kindly provides us with the data on the seven-day Eurodollar deposit rate. The literature has used all three

Table 2
Different proxies for the short rate

Data	Studies
Federal funds rate	Conley et al. (1997), a shorter sample
Seven-day Eurodollar rate	Ait-Sahalia, (1996a,b), Hong and Li (2005), Jones (2003)
Three-month Treasury yield	Stanton (1997) a shorter sample, Jiang (1998), Chapman and Pearson (2000), Durham (2003)

Entries summarize the studies that have used the three interest-rates series. These studies use either the whole or a sub-sample of the series that we use. Some studies use a lower sampling frequency.

series, or their shorter sample, in studying the short-rate dynamics. Table 2 provides a partial list of the studies that have used the three interest-rate series.

We estimate the models using the (quasi) maximum likelihood method. For the one-factor diffusion specification, we apply the following Euler approximation:

$$r_t - r_{t-\Delta} = \mu(r_{t-\Delta})\Delta + \sqrt{v(r_{t-\Delta})}\Delta\varepsilon_t, \tag{8}$$

where Δ denotes the length of the discrete interval ($\Delta = 1/252$ for our daily series) and ε_t denotes a standardized normal random variable with zero mean and unit variance. Stanton (1997) and Jones (2003) find that the bias from the Euler approximation is negligible when applied to daily frequency.

From the Euler approximation in Eq. (8), the log likelihood function becomes

$$\mathcal{L}(\{r_t\}_{t=2}^T) = -\frac{T-1}{2} \ln 2\pi - \frac{1}{2} \sum_{t=2}^T \left[\ln v(r_{t-\Delta})\Delta + \frac{(r_t - r_{t-\Delta} - \mu(r_{t-\Delta})\Delta)^2}{v(r_{t-\Delta})\Delta} \right], \tag{9}$$

where T denotes the number of observations. We ignore the unconditional (prior) likelihood of the first observation (r_1). Thus, strictly speaking, the estimation follows a quasi-maximum likelihood method.

For the GARCH-type models, based on the generalized error distribution assumption on the innovation, the log likelihood function is

$$\mathcal{L}(\{r_t\}_{t=2}^T) = (T-1) \ln \left[\frac{\theta}{\lambda 2^{(\theta+1)/\theta} \Gamma(1/\theta)} \right] - \frac{1}{2} \sum_{t=2}^T \left[\ln v_t + \left| \frac{r_t - r_{t-1} - \mu(r_{t-1})}{\lambda \sqrt{v_t}} \right|^\theta \right]. \tag{10}$$

We estimate the model parameters by maximizing the log likelihood functions.

5. Interest-rate dynamics

By comparing the parameter estimates and the maximized likelihood values under various restricted and unrestricted versions of the two model specifications and for different interest-rate series, we make inference on the interest-rate dynamics. We address the following questions: How does using different interest-rate series affect our conclusion on the short-rate dynamics? How does our conclusion on the nonlinearity

of the drift function vary under different specifications of the conditional variance function? How does incorporating GARCH-type volatility and non-normal innovation affect our conclusion on the drift specification? And finally, if there is nonlinearity, where does it come from?

5.1. Nonlinearity in the drift function of the one-factor diffusion models

Table 3 reports the parameter estimates and the *t*-statistics in parentheses for the one-factor diffusion models using the three interest-rate series. For each series, we

Table 3
Parameter estimates of the one-factor diffusion model

Drift	Federal funds rate			Seven-day Eurodollar rate			Three-month Treasury yield		
	General	Fifth	Affine	General	Fifth	Affine	General	Fifth	Affine
α_0	0.00304 (2.66)	0.00375 (5.68)	0.00032 (3.37)	-0.0325 (1.81)	0.0037 (2.27)	0.00014 (2.01)	0.00028 (0.64)	6e-5 (0.69)	2e-5 (1.64)
α_1	-0.2655 (3.60)	-0.3072 (5.63)	-0.00582 (3.45)	0.9291 (1.55)	-0.2567 (2.27)	-0.00115 (0.92)	-0.0198 (0.68)	-0.0055 (0.56)	-0.00032 (0.98)
α_2	7.8968 (4.01)	8.8980 (5.52)	-	-12.401 (1.26)	6.6287 (2.31)	-	0.6479 (0.75)	0.2420 (0.61)	-
α_3	-102.29 (4.20)	-113.43 (5.38)	-	80.996 (0.96)	-76.859 (2.35)	-	-10.406 (0.82)	-4.8288 (0.69)	-
α_4	584.09 (4.24)	640.85 (5.18)	-	-239.82 (0.67)	405.54 (2.36)	-	78.266 (0.88)	42.264 (0.75)	-
α_5	-1205.6 (4.20)	-1312.3 (4.98)	-	230.50 (0.39)	-792.11 (2.37)	-	-218.09 (0.94)	-131.24 (0.82)	-
α_6	3e-6 (0.60)	-	-	0.00042 (1.32)	-	-	-1e-6 (0.49)	-	-
β_0	3e-5 (2.59)	3e-5 (2.17)	3e-5 (2.35)	-3e-6 (22.4)	-3e-6 (21.0)	-3e-6 (22.2)	2e-7 (19.0)	2e-7 (19.2)	2e-7 (19.3)
β_1	0.0011 (17.2)	0.0011 (15.4)	0.0011 (18.2)	0.00011 (31.2)	0.00011 (29.6)	0.00011 (32.2)	-2e-6 (3.64)	-2e-6 (3.66)	-2e-6 (3.63)
β_2	-5e-5 (4.97)	-5e-5 (5.19)	-5e-5 (4.85)	0.2363 (3.66)	0.2145 (3.69)	0.2572 (3.77)	0.0198 (5.06)	0.0199 (5.10)	0.0203 (5.16)
β_3	0.1324 (2.31)	0.1393 (2.14)	0.1236 (2.30)	4.5329 (33.3)	4.4804 (33.1)	4.5645 (34.7)	3.8342 (44.8)	3.8367 (45.2)	3.8456 (45.8)
\mathcal{L}	45,699.60	45,698.89	45,674.53	24,352.71	24,351.61	24,346.44	67,152.67	67,152.49	67,151.76

Entries report the maximum likelihood estimates of the following one-factor diffusion model:

$$dr_t = \mu(r_t)dt + \sqrt{v(r_t)}dW_t, \tag{16}$$

with the following specification on the drift and variance functions,

$$\mu(r_t) = \alpha_0 + \alpha_1 r_t + \alpha_2 r_t^2 + \alpha_3 r_t^3 + \alpha_4 r_t^4 + \alpha_5 r_t^5 + \alpha_6 r_t^{-1}, \tag{17}$$

$$v(r_t) = \beta_0 + \beta_1 r_t + \beta_2 r_t^{\beta_3}. \tag{18}$$

We report the absolute magnitudes of the *t*-statistics in parentheses. We estimate the models using three interest-rate series: the federal funds rate, the seven-day Eurodollar rate, and the three-month Treasury yield. Under each interest-rate series, we report the estimates on the full, unrestricted model in the first column (titled “General”), a restricted version with $\alpha_6 = 0$ in the second column (titled “Fifth”), and a linear drift version with $\alpha_2 = \alpha_3 = \alpha_4 = \alpha_5 = \alpha_6 = 0$ in the third column (titled “Affine”).

report the maximum likelihood estimates of the unrestricted version of the model in the first column (“General”), a restricted version with $\alpha_6 = 0$ in the second column (“Fifth”), and an affine specification in the third column (“Affine”) with $\alpha_2 = \alpha_3 = \alpha_4 = \alpha_5 = \alpha_6 = 0$. With $\alpha_6 = 0$, the drift function is essentially a fifth-order polynomial function of the interest rate, thus the title “Fifth”. The drift specification under the title “Affine” constitutes a benchmark of affine drift dynamics.

We experience some identification problems in estimating the unrestricted version. Parameter estimates under this model have small t -values. Under all three interest-rate series, the estimates for α_6 , which control the r^{-1} term, are all very small and not significantly different from zero. When we set this parameter (α_6) to zero and re-estimate the model, the t -values of the other parameter estimates increase. The log likelihood values of this restricted version only decline slightly, indicating that a fifth-order polynomial is flexible enough to capture the drift function of all three interest-rate series. Therefore, henceforth we use the fifth-order polynomial drift specification as our general nonlinear specification and contrast it with the affine specification.

Comparing the t -statistics of the drift parameter estimates under the fifth-order polynomial specification for the three interest-rate series, we observe that the significance of the nonlinear terms (α_2 to α_5) depends crucially on the choice of data sets. When using either the federal funds rate or the seven-day Eurodollar rate, the estimates for all the drift parameters, from α_0 to α_5 , are statistically significant. Therefore, we identify strong nonlinearity in the drift function for these two interest-rate series under the one-factor diffusion specification.

In contrast, none of the drift function parameter estimates are significant for the three-month Treasury yield. Even the coefficients on the constant and linear terms are not significantly different from zero. In other words, our estimates suggest that the drift dynamics of the three-month Treasury yield is not significantly different from a random walk assumption.

Under the third column (“Affine”) for each interest-rate series, we also estimate a affine drift version of the model. We can construct a log likelihood ratio test between the fifth-order polynomial specification and the affine specification. If we let $\Delta\mathcal{L}$ denote the log likelihood difference between the fifth-order polynomial specification and the affine specification, the likelihood ratio statistic, $LR = 2\Delta\mathcal{L}$, has a Chi-square distribution with four degrees of freedom, under the null hypothesis that the drift function is affine: $\alpha_2 = \alpha_3 = \alpha_4 = \alpha_5 = 0$. The critical value for this Chi-square test is 9.49 at the 95% confidence level. The estimated LR statistics are 48.72, 10.34, and 1.46 for the federal funds rate, the seven-day Eurodollar rate, and the three-month Treasury yield, respectively. Therefore, under the one-factor diffusion specification, the LR tests strongly reject the null hypothesis of linear drift dynamics for the federal funds rate and the seven-day Eurodollar rate, but cannot reject the linear-drift hypothesis for the three-month Treasury yield.

Both the model parameter estimates and the likelihood ratio tests show that the conclusion on the short-rate drift dynamics can either be highly nonlinear or no different from a random walk, depending on the choice of the interest-rate series.

5.2. Nonlinearity in the variance function and its impact on the drift specification

Table 3 also contains the parameter estimates on the variance function under different interest-rate series and different restrictions on the drift specification. All the variance function parameters are strongly significant for all three interest-rate series. The parameter estimates vary greatly with the choice of different interest-rate series, but do not vary much with different restrictions on the drift specifications.

The significant parameter estimates for the variance function indicate that the conditional variance of the interest-rate changes is a nonlinear function of the interest-rate level. Either the affine specification or the CEV specification alone cannot fully capture this nonlinearity.

To test the significance of the combined form against either affine or CEV variance specification, we estimate models with an affine variance function and a CEV variance function separately. To save space, we do not report the parameter estimates of the restricted versions, although they are available upon request. We only report the maximized log likelihood values of these restricted models in Table 4.

For each interest-rate series, Panel A of Table 4 reports the maximized log likelihood values of six models, a multiplication of two drift specifications and three variance function specifications. The two drift functions are the affine form and the fifth-order polynomial specification:

$$\begin{aligned} \text{Affine} \quad \mu(r_t) &= \alpha_0 + \alpha_1 r_t, \\ \text{Fifth} \quad \mu(r_t) &= \alpha_0 + \alpha_1 r_t + \alpha_2 r_t^2 + \alpha_3 r_t^3 + \alpha_4 r_t^4 + \alpha_5 r_t^5. \end{aligned} \quad (11)$$

The three variance function specifications include the affine, the CEV, and the combined form:

$$\begin{aligned} \text{Affine} \quad v(r_t) &= \beta_0 + \beta_1 r_t, \\ \text{CEV} \quad v(r_t) &= \beta_2 r_t^{\beta_3}, \\ \text{Combined} \quad v(r_t) &= \beta_0 + \beta_1 r_t + \beta_2 r_t^{\beta_3}. \end{aligned} \quad (12)$$

To test the null hypothesis of affine or CEV variance specification against the combined form, we form the log likelihood ratio tests as discussed before and report the statistics in Panel B of Table 4. Under both the null hypothesis of affine variance and the null hypothesis of CEV variance, the LR statistics have a Chi-square distribution with two degrees of freedom. The 95% and 99% confidence level critical values for this statistic are 5.99 and 9.21, respectively. The LR statistics are all well above these critical values for all three interest-rate series and for both drift specifications. These test results indicate a strong rejection of the affine and CEV variance-function specifications in favor of the combined form.

Previous studies have tested the drift dynamics under different specifications of the variance function. To investigate how different specifications of the variance function affect the conclusions on the drift dynamics, we report in Panel C of Table 4 the likelihood ratio statistics from testing the null hypothesis of linear drift against the alternative of a fifth-order polynomial specification, under different variance specifications and different interest-rate series. Again, these LR statistics have a Chi-square

Table 4
Log likelihood values of one-factor diffusion models

Drift	Federal funds rate		Seven-day Eurodollar rate		Three-month Treasury yield	
	Fifth	Affine	Fifth	Affine	Fifth	Affine
<i>A. The log likelihood values, \mathcal{L}</i>						
Affine	45,122.03	45,047.94	24,083.53	24,054.07	65,880.79	65,870.53
CEV	44,902.49	44,818.67	24,326.55	24,317.88	66,293.38	66,287.46
Combined	45,698.89	45,674.53	24,351.61	24,346.44	67,152.49	67,151.76
<i>B. Variance LR = $2(\mathcal{L}_{Combined} - \mathcal{L}_{Affine/CEV}) \sim \chi^2(2)$</i>						
Affine	1,592.8	1,711.72	50.12	57.12	1,718.22	1,728.6
CEV	1,153.72	1,253.18	536.16	584.74	2,543.4	2,562.46
<i>C. Drift LR = $2(\mathcal{L}_{Fifth} - \mathcal{L}_{Affine}) \sim \chi^2(4)$</i>						
Affine		148.18		58.92		20.52
CEV		167.64		17.34		11.84
Combined		48.72		10.34		1.46

Entries report the maximized log likelihood values for different versions of the one-factor diffusion model. We estimate six versions of the one-factor diffusion model for each interest-rate series, resulting from a combination of two drift function specifications and three variance function specifications. The two drift specifications are:

$$\text{Affine } \mu(r_t) = \alpha_0 + \alpha_1 r_t,$$

$$\text{Fifth } \mu(r_t) = \alpha_0 + \alpha_1 r_t + \alpha_2 r_t^2 + \alpha_3 r_t^3 + \alpha_4 r_t^4 + \alpha_5 r_t^5.$$

The three variance function specifications are:

$$\text{Affine } v(r_t) = \beta_0 + \beta_1 r_t,$$

$$\text{CEV } v(r_t) = \beta_2 r_t^{\beta_3},$$

$$\text{Combined } v(r_t) = \beta_0 + \beta_1 r_t + \beta_2 r_t^{\beta_3}.$$

The first panel reports the maximized log likelihood value for each model. The second panel reports the likelihood ratio (LR) statistics from testing the null hypothesis of affine or CEV variance specification against the combined specification. The LR statistics has a Chi-square distribution with two degrees of freedom. The 95% and 99% critical values are 5.99 and 9.21, respectively. The third panel reports the LR statistics from testing the null hypothesis of linear drift specification against the fifth-order polynomial specification. The LR statistics has a Chi-square distribution with four degrees of freedom. The 95% and 99% critical values are 9.49 and 13.28, respectively.

distribution with four degrees of freedom. The 95% and 99% confidence level critical values are 9.49 and 13.28, respectively.

For each of the three variance function specifications, the LR statistic becomes smaller as the maturity of the interest-rate series increases from the overnight federal funds rate, to the seven-day Eurodollar rate, and to the three-month Treasury yield. Therefore, the finding of nonlinearity is sensitive to the choice of data sets under all three specifications of the variance function.

For the same interest-rate series, the variance function specification also has an obvious effect on the LR statistics on the drift dynamics. The statistic becomes larger, and thus the evidence on drift nonlinearity becomes stronger, if we use an affine variance function or a CEV function, rather than the combined form. Under the

affine variance specification, the LR statistics reject the null hypothesis of a linear drift at the 99% confidence level for all three interest-rate series. Under the CEV specification, the LR statistics reject the linear-drift hypothesis at the 99% confidence level for the federal funds rate and the seven-day Eurodollar rate, and at the 95% confidence level for the three-month Treasury yield. When the variance function takes the most flexible combined form, the linear-drift hypothesis is rejected at the 99% confidence level only for the federal funds rate. For the seven-day Eurodollar rate, the LR statistic rejects the linear-drift hypothesis at the 95% confidence level. For the three-month Treasury yield, the LR statistic can no longer reject the null hypothesis of linear drift.

The above analysis shows why the existing evidence on the short-rate dynamics is so confusing. The conclusion on the drift dynamics depends not only on the choice of the particular data set, but also on the specification of the variance function. The variations in these two choices generate conflicting conclusions in the previous studies.

5.3. Impacts of GARCH volatility and non-normal innovation

The fact that the conditional variance specification significantly affects the conclusion on the drift dynamics makes us wonder whether the identified nonlinear drift is truly due to nonlinearity in the drift function or due to misspecification in the interest-rate innovation component. In this subsection, we investigate how much our conclusions on the drift dynamics vary when we incorporate both GARCH volatility and non-normal innovation in the interest-rate dynamics.

To save space, we do not report the parameter estimates of the new models, but only report the maximized log likelihood values and the results on various likelihood ratio tests. We summarize these results in Table 5. Panel A in Table 5 reports the maximized log likelihood values for each model. We estimate four models for each interest-rate series, a multiplication of two conditional mean (drift) specifications with two GARCH-type conditional variance specifications. The two drift functions are of the linear form and the fifth-order polynomial specification as in Eq. (11). The two GARCH variance specifications are

$$\begin{aligned} \text{GARCH-Affine} \quad v_t &= \beta_0 + \beta_1 \varepsilon_{t-1}^2 + \beta_2 v_{t-1} + \beta_3 r_{t-1}, \\ \text{GARCH-CEV} \quad v_t &= h_t r_{t-1}^{\beta_3}, \text{ with } h_t = \beta_0 + \beta_1 \varepsilon_{t-1}^2 + \beta_2 h_{t-1}, \end{aligned} \quad (13)$$

labeled as GARCH-Affine and GARCH-CEV, respectively.

Panel B of Table 5 constructs the likelihood ratio statistics, testing the null hypothesis of linear drift against the alternative of fifth-order polynomial drift, under the two GARCH specifications for the conditional variance. Both GARCH specifications generate similar results. The LR statistics can no longer reject the null hypothesis of linear drift for both the seven-day Eurodollar rate and the three-month Treasury yield.

The nonlinearity in the drift function of the federal funds rate remains strong, despite the incorporation of the GARCH effects and non-normal innovations. Therefore, using a flexible diffusion function, incorporating GARCH volatility, and

Table 5
Log likelihood values of GED-GARCH models

Drift	Federal funds rate		Seven-day Eurodollar rate		Three-month Treasury yield	
	Fifth	Affine	Fifth	Affine	Fifth	Affine
<i>A. The log likelihood values, \mathcal{L}</i>						
GARCH-Affine	59,297.96	59,273.29	31,015.76	31,014.98	79,843.47	79,840.3
GARCH-CEV	59,279.89	59,245.33	31,050.79	31,049.28	79,802.55	79,798.66
<i>C. Drift LR = $2(\mathcal{L}_{Fifth} - \mathcal{L}_{Affine}) \sim \chi^2(4)$</i>						
GARCH-Affine	49.34		1.56		6.34	
GARCH-CEV	69.12		3.02		7.78	

Entries report the maximized log likelihood values for different versions of GED-GARCH models. We estimate four models for each interest rate series, resulting from a combination of two drift function specifications and two GARCH specifications. The two drift specifications are:

Affine $\mu(r_t) = \alpha_0 + \alpha_1 r_t,$

Fifth $\mu(r_t) = \alpha_0 + \alpha_1 r_t + \alpha_2 r_t^2 + \alpha_3 r_t^3 + \alpha_4 r_t^4 + \alpha_5 r_t^5.$

The two GARCH specifications are:

GARCH-Affine $v_t = \beta_0 + \beta_1 \varepsilon_{t-1}^2 + \beta_2 v_{t-1} + \beta_3 r_{t-1},$

GARCH-CEV $v_t = h_t r_{t-1}^{\beta_3},$ with $h_t = \beta_0 + \beta_1 \varepsilon_{t-1}^2 + \beta_2 h_{t-1},$

where v_t denotes the conditional variance of interest-rate changes, $v_t = E_{t-1}[\varepsilon_t^2],$ with $\varepsilon_t = r_t - E_{t-1}[r_t].$ The log likelihood for the model is based on an assumption of a generalized error distribution (GED) on the error term. The first panel reports the maximized log likelihood value for each model. The second panel reports the LR statistics from testing the null hypothesis of linear drift against the fifth-order polynomial drift specification. The LR statistics has a Chi-square distribution with four degrees of freedom. The 95% and 99% critical values are 9.49 and 13.28, respectively.

specifying non-normal innovation can reduce the statistical significance of nonlinearity in the drift function of some interest-rate series, but these added flexibilities on the conditional variance and innovation specification cannot fully account for the nonlinearity observed in the drift dynamics of the overnight federal funds rate.

5.4. Sources of nonlinearity

The statistical significance of nonlinearity in the short-rate drift depends not only on the data choice, but also on the specifications of the conditional variance and innovation distribution. The nonlinearity in the drift of the three-month Treasury yield is weak. Its statistical significance disappears when we use a flexible conditional variance specification. In contrast, under all the variations in the parametric specifications, the nonlinearity in the drift of the overnight federal funds rate remains strong and statistically significant. An important question to ask is: Where does the strong nonlinearity in the overnight federal funds rate come from? Why is the nonlinearity in the three-month Treasury yield so much weaker?

The literature has cited institutional differences for the different behaviors. For example, Duffee (1996) finds that there is substantial idiosyncratic variation in the one-month Treasury-bill yields. Hamilton (1996) documents a systematic variation

in the federal funds rate due to the settlement effect in federal funds market. Here, we look more carefully into the daily time series of the data in identifying the source of nonlinearity.

For each interest-rate series, we compute the maximized log likelihood values for different models at each date. We focus on two one-factor diffusion specifications: the affine drift specification and the fifth-order polynomial specification, both having the general, combined form for the variance function. Then, we compute the daily likelihood ratio statistics (LR_t) between two models, defined as twice the difference between the daily log likelihood values of the two models. Summation of these daily likelihood ratios over the whole sample generates the likelihood ratio test statistics for the null hypothesis of linear drift against the alternative of fifth-order polynomial nonlinear drift. By analyzing the likelihood ratio at a daily level, we investigate at which days the two models differ the most. Then, we identify the unique features of these days to understand the source of interest-rate nonlinearity.

Fig. 1 plots the daily likelihood ratio statistics for the three interest-rate series. For ease of comparison, we apply the same scale on the *x*-axis so that the dates

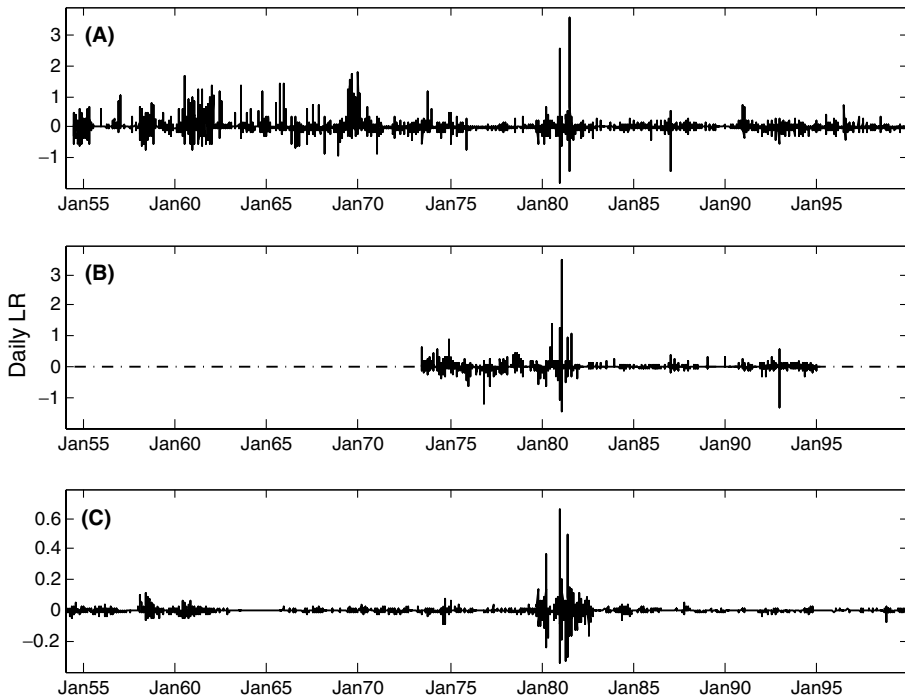


Fig. 1. Time series of daily log likelihood ratio statistics. Solid lines denote the daily likelihood ratio statistics under the null hypothesis of linear drift against the alternative of fifth-order polynomial specification, under the one-factor diffusion framework and the general variance function specification. The three panels correspond to the three interest-rate series: (A) the federal funds rate, (B) the seven-day Eurodollar deposit rate, and (C) the three-month Treasury yield.

for the three panels align with one another. We also use the same scale for the y-axis for the federal funds rate and the seven-day Eurodollar deposit rate, but we choose to a smaller scale for the three-month Treasury yield (about one-fifth of the scale for the other two interest-rate series) to reflect the smaller estimates for the likelihood ratios on this series.

The likelihood ratio statistic is positive when the nonlinear drift specification performs better than the linear drift specification. By aligning the dates in the three panels in Fig. 1, we immediately identify the main source of nonlinearity in the drift specification. The highest daily likelihood ratio estimates for all three interest-series occur around the early 1980s. The magnitudes of the daily log likelihood ratios are similar for the federal funds rate and the seven-day Eurodollar deposit rate, but smaller for the three-month Treasury yield. The daily likelihood ratios for the three interest-rate series show similar qualitative patterns across time.

Fig. 2 plots and compares the time series of the three interest-rate series. We again align the dates for the three panels. We find that the three interest-rate series reach their respective historical high levels during the early 1980s. There have been significant regime shifts in the way that Federal Reserve handled the money supply and interest rates during the sample period considered in this paper. Specifically, the October 1979–September 1982 period witnessed the Federal Reserve's experiment.

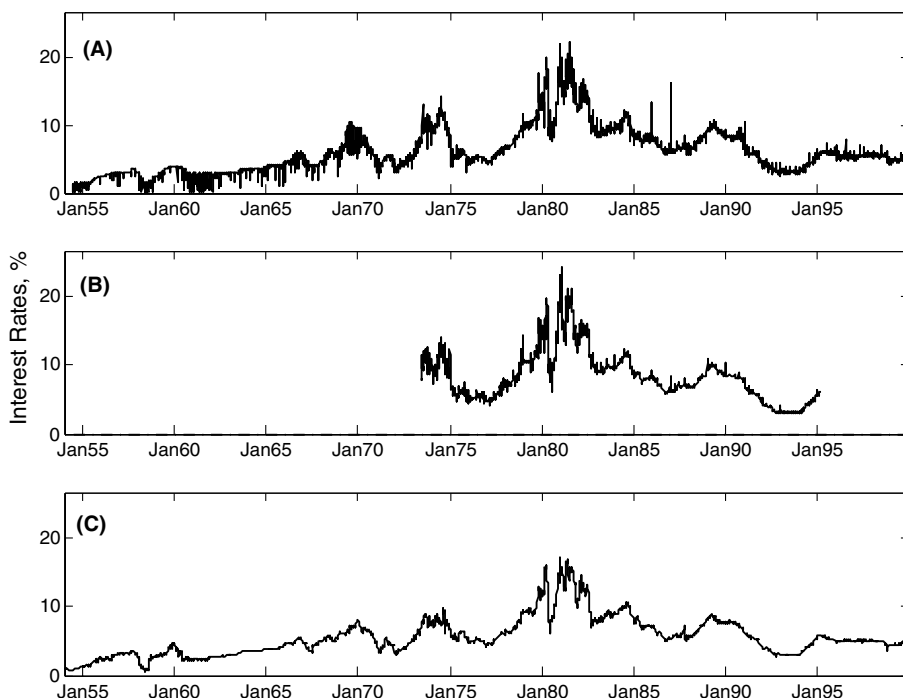


Fig. 2. Interest-rate time series. Lines denote the time series of (A) the federal funds rate, (B) the seven-day Eurodollar deposit rate, and (C) the three-month Treasury yield.

The operating target of monetary policy became the amount of non-borrowed reserves with the banking system, and both the level and the volatility of interest rates reached levels never experienced before. This period reflects the surprise shift in monetary policy on October 6, 1979, when Paul Volker, then new chairman of the Federal Reserve, called an extraordinary Saturday meeting of the Federal Reserve Open Market Committee to obtain approval for a tight monetary policy to fight inflation.

Analysis of the historical interest-rate level and the daily likelihood ratio statistics indicate that when the short-term interest-rate level reaches historical highs, the market expects a faster-than-usual speed of mean-reversion. It is during these extreme cases where a simple affine drift specification becomes no longer adequate in capturing the behavior of the interest rates.

Another way to pool the information in Figs. 1 and 2 together is to plot the daily likelihood ratio against the previous day's interest-rate level. From such a plot, we are asking: Under what interest-rate levels do we expect a fifth-order polynomial specification to capture the interest-rate movements better than the linear drift specification? Fig. 3 shows such conditional likelihood ratio plots for the three interest-rate series. The log likelihood ratio statistic becomes strongly positive and hence

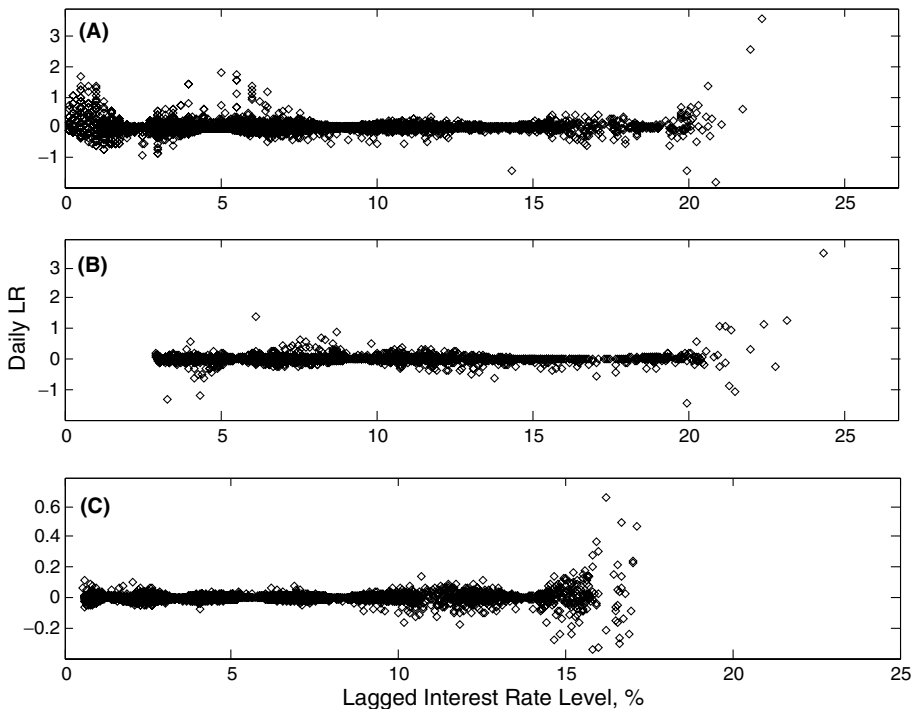


Fig. 3. Daily log likelihood ratio statistics at different interest levels. The scatter diagrams plot the likelihood ratio statistics against the lagged interest-rate level for (A) the federal funds rate, (B) the seven-day Eurodollar deposit rate, and (C) the three-month Treasury yield.

favors a nonlinear drift specification only at extremely high interest-rate levels (the right corners).

By aligning the three panels with the same interest-rate level scale, we observe that the historical highs for the federal funds rate and the seven-day Eurodollar deposit rate are much higher than the historical highs for the three-month Treasury yield. The maximum level for the three-month Treasury yield in our sample is 17.14%, but the maxima for the federal funds rate and the seven-day Eurodollar deposit rate are 22.36% and 24.33%, respectively.

With an upward-sloping mean term structure, the three-month Treasury yield is on average higher than the federal funds rate. When the short-term rate reaches historical highs, the market expects the rate to come back quickly to some long-run level by virtue of stronger-than-normal mean reversion. Longer-term interest rates take such expectation into account and stay at a relatively low level, thus generating a reversed downward-sloping term structure. The three-month Treasury yield takes part of this expectation into account and becomes lower than the overnight federal funds rate during the historical highs of the early 1980s.

Fig. 4 illustrates this point by plotting the time series of the difference between the three-month Treasury yield and the overnight federal funds rate during their overlapping sample periods. At the start of the sample when the interest-rate levels are low, three-month Treasury yields are mostly higher than the federal funds rates, generating positive differences. But as the interest-rate level rises, the difference becomes negative. The differences are particularly negative during the two interest-rate spikes in the mid 1970s and in the early 1980s.

At most times, a linear drift specification is sufficient to capture the persistence and mean reversion of interest-rate movements. But when the short-term interest rate reaches historical highs, the federal reserve begins to take dramatic measures

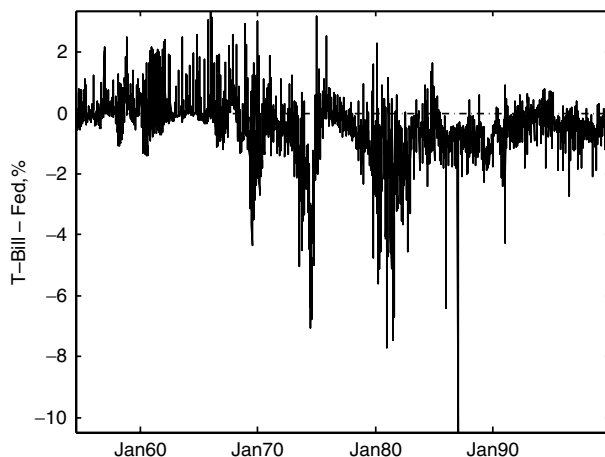


Fig. 4. The time series of the differences between the three-month Treasury yield and federal funds rate. The solid line plots the difference in percentage points between the three-month Treasury yield and the federal funds rate.

to reign in inflation. As a result, the short-term interest rate falls shortly after, generating a higher-than-normal speed of mean-reversion. The difference in the mean-reversion speeds at different interest-rate levels produces nonlinearity in the short-rate drift function.

When the short-term interest rate reaches historical highs, the market expects a faster mean reversion than experienced at other times. This expectation reflects itself in longer-maturity interest rates. As a result, the longer-maturity interest rates stay at a relatively low level at historical highs and stay at a relatively high level at historical lows, thus partially negating the difference in mean reversion speeds at different time periods. As a result, the switch of mean-reversion speeds at different time periods and hence nonlinearity is not as evident in longer-term interest rates such as the three-month Treasury yield as in the very short-term rates such as the overnight federal funds rate.

6. Stochastic volatility

We have so far tested the presence and significance of nonlinearity in the short-rate drift using a continuous-time one-factor diffusion framework and a discrete-time GED-GARCH model. We estimate the continuous-time one-factor specifications as a synthesis of the literature. We estimate the GED-GARCH models to study the impacts of time-varying volatility and discontinuous interest-rate movements. Nelson (1990) shows that the continuous-time limit of a GARCH(1,1)-specification is a stochastic volatility model.¹ As a robustness check, this section estimates a class of continuous-time two-factor interest-rate models:

$$dr_t = (\alpha_0 + \alpha_1 r_t + \alpha_2 r_t^2 + \alpha_3 r_t^3 + \alpha_4 r_t^4 + \alpha_5 r_t^5 + \alpha_6 r_t^{-1})dt + \sqrt{v_t} dW_{1t}, \quad (14)$$

$$d \ln v_t = (\beta_0 + \beta_1 \ln v_t)dt + \omega dW_{2t}, \quad (15)$$

where the two standard Brownian motions W_{1t} and W_{2t} are correlated: $\rho dt = E(dW_{1t}dW_{2t})$.

We estimate various restricted and unrestricted versions of the above model specification using the quasi-maximum likelihood method of Harvey (1989) and Kim et al. (1998). Table 6 presents the parameter estimates, t -statistics, and maximized log-likelihood values. For all interest-rate series and for all drift specifications, we find strong mean reversion in instantaneous variance dynamics. The maximum likelihood estimates of β_1 are negative and highly significant in all cases. The estimates for the volatility of volatility coefficient ω are also statistically significant, implying that the volatility of the interest-rate changes is by itself volatile.

The parameter estimates for the drift functions in Table 6 provide supporting evidence for our earlier findings from the GED-GARCH model. For the three-month Treasury bill and seven-day Eurodollar rates, the t -statistics of α_2 , α_3 , α_4 , α_5 and α_6

¹ Alexander and Lazar (2005) derive the continuous-time limit of more general GARCH-type models.

Table 6
Parameter estimates of the stochastic volatility models

	Federal funds rate			Seven-day Eurodollar rate			Three-month Treasury yield		
	General	Fifth	Affine	General	Fifth	Affine	General	Fifth	Affine
α_0	0.0036 (2.57)	0.0039 (5.72)	0.00051 (4.41)	0.0453 (1.93)	0.0063 (2.01)	0.00042 (1.96)	0.0007 (1.10)	0.0002 (1.43)	0.00043 (1.72)
α_1	-0.2871 (3.36)	-0.3164 (5.57)	-0.0086 (4.47)	0.8527 (1.25)	-0.8952 (1.43)	-0.0058 (2.72)	-0.0562 (1.20)	-0.0302 (1.29)	-0.00069 (-0.88)
α_2	7.6878 (3.56)	8.5678 (5.45)	-	11.093 (1.49)	10.894 (1.44)	-	1.7052 (1.24)	1.1081 (1.28)	-
α_3	-104.92 (3.91)	-115.55 (5.43)	-	124.02 (1.70)	-120.32 (1.67)	-	-24.671 (1.29)	-19.842 (1.27)	-
α_4	598.43 (4.13)	635.52 (5.5)	-	602.33 (1.65)	614.03 (1.71)	-	180.34 (1.22)	130.73 (1.35)	-
α_5	-1217.62 (4.51)	-1268.01 (5.61)	-	-1145.01 (1.58)	-1132.01 (1.47)	-	-478.09 (1.24)	-387.4 (1.31)	-
α_6	1e-6 (0.43)	-	-	-0.0004 (0.91)	-	-	-3e-6 (0.74)	-	-
β_0	-7.6601 (27.75)	-7.7196 (28.36)	-8.9843 (34.89)	-8.9255 (22.51)	-8.8342 (21.92)	-8.497 (19.77)	-11.284 (30.41)	-11.501 (30.67)	-11.892 (28.80)
β_1	-0.5696 (26.45)	-0.575 (27.04)	-0.6671 (36.09)	-0.6028 (24.01)	-0.5819 (23.12)	-0.5799 (20.84)	-0.7001 (32.89)	-0.72 (33.09)	-0.6892 (30.13)
ω	3.1271 (48.98)	3.0819 (46.53)	2.761 (57.00)	2.9341 (47.10)	2.9017 (44.80)	3.1482 (48.91)	2.8762 (50.81)	2.966 (51.00)	2.8592 (52.38)
ρ	-0.00049 (2.43)	-0.00048 (2.40)	-0.00034 (2.29)	3.1e-5 (1.12)	3.7e-5 (1.25)	2.2e-5 (0.94)	6.1e-5 (1.85)	6.2e-5 (1.97)	7.0e-5 (1.87)
\mathcal{L}	58,445.42	58,445.26	58,423.19	29,879.56	29,879.22	29,878.11	75,783.92	75,783.64	75,781.79

Entries report the maximum likelihood estimates of the following two-factor diffusion model:

$$dr_t = (\alpha_0 + \alpha_1 r_t + \alpha_2 r_t^2 + \alpha_3 r_t^3 + \alpha_4 r_t^4 + \alpha_5 r_t^5 + \alpha_6 r_t^{-1})dt + \sqrt{v_t}dW_{1t},$$

$$d \ln v_t = (\beta_0 + \beta_1 \ln v_t)dt + \omega dW_{2t},$$

where the two standard Brownian motions W_{1t} and W_{2t} are correlated: $\rho dt = E(dW_{1t}, dW_{2t})$. We report the absolute magnitudes of the t -statistics in parentheses. We estimate the models using three interest-rate series: the federal funds rate, the seven-day Eurodollar rate, and the three-month Treasury yield. Under each interest-rate series, we report the estimates on the full, unrestricted model in the first column (titled “General”), a restricted version with $\alpha_6 = 0$ in the second column (titled “Fifth”), and a linear drift version with $\alpha_2 = \alpha_3 = \alpha_4 = \alpha_5 = \alpha_6 = 0$ in the third column (titled “Affine”).

are very low, and not even significant at the 10% level in most cases. Furthermore, the log likelihood difference between the linear drift specification and the fifth-order drift specification is small, hence showing little evidence of nonlinearity in the conditional mean of interest-rate changes.

In contrast, for the federal funds rate, the estimated nonlinearity in the drift function is strong, despite the incorporation of the stochastic volatility factor. The estimates for the nonlinearity parameters α_2 , α_3 , α_4 , α_5 and α_6 are all statistically significant at the 1% level. The maximized likelihood values for the linear and the fifth-order drift specifications also suggest a strong rejection of the linear drift specification in favor of the nonlinear specification.

7. Conclusions

The dynamics of the short-term interest rate constitute the key building block in asset pricing. The literature presents conflicting evidence on the exact dynamics of the interest-rate process, particularly regarding the nonlinearity of the conditional mean of interest-rate changes as a function of the lagged interest-rate level. The conflicting evidence is partially due to the use of different data sets as a proxy for the short rate and the use of different parametric/nonparametric specifications under which the studies perform the statistical tests. In this paper, we provide a comprehensive analysis of the interest-rate dynamics by considering three different data sets and two flexible parametric specifications.

Of the two parametric specifications, we propose a flexible one-factor diffusion framework that encompasses most parametric specifications in the literature. We also consider GARCH-type models with non-normal innovations to capture the potential impact of time-varying volatility and discontinuous interest-rate movements. We estimate both sets of models on the three interest-rate series using the quasi-maximum likelihood estimation method.

We find that nonlinearities are strong in the federal funds rate and the seven-day Eurodollar rate, but are much weaker in the three-month Treasury yield. When the model specification allows for both GARCH volatility and non-normal interest-rate innovation, likelihood ratio tests can no longer reject a linear drift specification for the three-month Treasury yield and the seven-day Eurodollar deposit rate, but the linear specification is strongly rejected for the overnight federal funds rate. We obtain similar findings when we estimate a two-factor diffusion model with stochastic volatility.

To understand the source of nonlinearity and the reasons behind different conclusions drawn on different interest-rate series, we analyze the daily log likelihood ratio statistics for the three interest-rate series between a fifth-order polynomial and a linear drift specification. We find that the likelihood ratios are close to zero at most times, but become strongly positive when the interest rates reach historical highs during the early 1980s. We conclude that the speeds of mean-reversion for short-term rates are different at normal times than at extremely high interest-rate era when the Federal Reserve is more likely to take drastic measures to aggressively fight inflation. The difference in mean-reversion speeds at different interest-rate levels generates the nonlinearity in the short-rate drift.

All three interest-rate series exhibit the same pattern for the likelihood ratio statistics, but the magnitude of the likelihood ratio for the three-month Treasury yield is smaller, resulting in statistical insignificance. By comparing the levels of the overnight federal funds rate and the three-month Treasury yield, we find that the Treasury yield is higher than the federal funds rate at low short-rate levels, but is lower than the federal funds rate at high short-rate levels. Such reversals on the term structure reflect the market expectation on future mean reversions in the interest-rate movements. The incorporation of the market expectation on the longer-maturity Treasury yield partially negates the difference in mean-reversion speeds to the point where the likelihood ratio tests cannot statistically reject the null hypothesis of an

affine drift specification. Thus, we conclude that nonlinearity exists in the very short-term interest rates due to different speeds of mean reversion at different interest-rate levels. This difference becomes smaller for longer-maturity interest rates due to the smoothing effect of market expectation. As a result, it is more difficult to identify nonlinearities in the longer-maturity interest rates than in the very short-term interest rates.

Our findings have important implications for both interest-rate modeling and forecasting. For modeling, our findings show that we need to either incorporate nonlinear drift specification to the short rate process, or apply a regime-switching framework, where each regime can have an affine specification with different mean-reversion speeds. For forecasting, our results indicate that incorporating nonlinearity is more likely to improve the predictive power of the models at extremely high interest-rate levels than at normal times.

Acknowledgments

We thank Giorgio P. Szegö (the editor), two anonymous referees, Ozgur Demirtas, Bin Gao, Susan Ji, Salih Neftci, Neil Pearson, Jonathan Wang, and seminar participants at the 2001 Financial Management and Southern Finance Association Meetings for comments. Financial support from the Eugene Lang Research Foundation of Baruch College is gratefully acknowledged.

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