Simple Robust Hedging with Nearby Contracts

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Archimedes-style hedging

*Give me a lever long enough and a fulcrum on which to place it, and I shall move the world.*

— Archimedes, Mathematician and inventor of ancient Greece, 287-212BC

In hedging derivatives risk, many think like Archimedes, by making strong, idealistic assumptions on the security dynamics and trading environment.

- Black-Merton-Scholes (1973) introduce the dynamic hedging concept, by assuming
  - The underlying security follows a one-factor diffusion process.
  - One can rebalance the hedge portfolio continuously.
- Carr and Wu (2002) propose a static hedge on vanilla options, by assuming
  - The underlying security follows a one-factor Markovian process.
  - One can deploy an infinite number of short-term options across the whole continuum of strikes.
In reality, the lever is not quite as long

- Transaction cost is a fact of life. Both continuous rebalancing and transacting on a continuum of options lead to financial ruin.
  - Discretization is a must.

- One does not know the exact dynamics of the underlying security: (i) how many risk sources and (ii) what the risk exposures are.
  - Riskfree hedge under model A is not riskfree under model B.
  - Neither hedge is riskfree in reality.

- Archimedes (or an Archimedes-style hedger) does not care, as all he wants is to guarantee that he’s absolutely right, under his own conditions.

- Practitioners do care, as they want to be approximately, relatively right (so that they do not lose their shirts), under all conceivable conditions.

- In this paper, we do not intend to prove our own righteousness under our own conditions, but strive to solve the practitioner’s problem in being about right in all scenarios.
Practitioners’ remedy

- Use BMS model to perform both delta and vega hedge.
  - Delta is balanced daily.
  - Vega and stress risk are managed opportunistically.

- Acknowledge that this vega is not that vega.
  - Vega at different strike and maturity ranges are different types of vegas.

- Example I: A portfolio with long $10 million vega at 5-year at-the-money and short $10 million vega at 4-year at-the-money is treated as relatively vega-risk free — vega is netted.

- Example II: A portfolio with long $10 million vega at 5-year at-the-money and short $10 million vega at 1-month 10-delta put is treated as having significant ($20 million) vega exposure — vega is added.

- From academic perspective, stochastic volatility can be generated from diffusion risk or jump risk, market risk or credit risk, short-term shock or long-term trend shift...
Our approach: Hedging with nearby contracts

- Academics often worry about the exact risk exposure calculation: Should the delta be calculated under a local vol model or a model with jump?
  - With the right exposure estimate, one can pick any contracts to neutralize the risk by solving a system of equations.
- Traders are less pedantic about the calculation, but intuitively realize that achieving vega (duration, etc) neutrality with nearby contracts is safer than using contracts that are far apart (in maturity and/or strike).
- We do not attempt to eliminate risks completely under a hypothetical model.
- We devise a simple robust hedging strategy that limit losses under all possible scenarios, regardless of model assumptions.
  - We do not assume a model, nor do we calculate risk exposures.
  - We design the hedging portfolio based on affinities in contract characteristics (such as strike and maturity).
    — As long as we hedge a target with a similar, nearby contract, the risk exposure cannot be too big.
Hedge with a maturity-strike triangle

- We hedge a target vanilla call option at strike $K$ and expiry $T$, $C(K, T)$, with three nearby call option contracts:
  - Using more than three contracts to hedge is not practical given transaction cost — Forget about a continuum.
  - $K_d < K_c < K_u$, with $K \in (K_d, K_u)$ and ideally $K_c = K$ when available.
- No theoretical constraints on the maturities, but only practical considerations:
  - Since often fewer maturities are available than strikes, we focus on two maturities instead of three, with $K_c$ at one maturity $T_c$ and $(K_d, K_u)$ at another maturity $T_o$ to form a maturity-strike triangle.
  - Since short-term options tend to be more liquid than long-term options, we might need to choose $(T_c, T_o) < T$ in practice.
Assumptions

- We assume that there are a finite number of (at least 3) options for us to choose to form the hedge portfolio and to compute a local volatility,
  
  \[ \sigma^2(K, T_o) = \frac{2C_T(K, T_o)}{C_{KK}(K, T_o)}. \]

- The concept of local volatility is originally developed by Dupire (94) under a one-factor diffusion setting.

- Positive local volatility exists under a much more general setting.

- We are not concerned with the dynamics, but rather try to obtain a stable estimate of the relation via interpolations and extrapolations:
  
  - Local quadratic fitting along the strike dimension on BMS implied volatilities to obtain \( IV_k \) and \( IV_{kk} \) estimates, \( k = \ln K \).
  
  - Local linear fitting along the maturity dimension on BMS implied volatilities to obtain a \( IV_T \) estimate.
  
  - Set the bandwidth large enough to ensure a positive, smooth local volatility surface.
    
    \[ \sigma^2 = \frac{2TII_T + I^2}{(1 - kl_k / I)^2 + TII_{kk} - \frac{1}{4} T^2 I^2 I_{kk}}. \]
Deriving portfolio weights on the maturity-strike triangle

The portfolio weights are obtained via the following steps:

1. Taylor expand both target and hedge options around \((K, T_o)\) to first order in \(T\) and second-order in \(K\):

\[
\begin{align*}
C(K, T) & \approx C + C_T(T - T_o), \\
C(K_d, T_o) & \approx C + C_K(K_d - K) + \frac{1}{2} C_{KK}(K_d - K)^2, \\
C(K_u, T_o) & \approx C + C_K(K_u - K) + \frac{1}{2} C_{KK}(K_u - K)^2, \\
C(K_c, T_c) & \approx C + C_K(K_c - K) + C_T(K, T_o)(T_c - T_o) + \frac{1}{2} C_{KK}(K_c - K)^2.
\end{align*}
\]

2. Replace \(C_{KK}\) with \(C_T\) via the local volatility definition.

3. Choose the three weights \((w_d, w_c, w_u)\) to match coefficients on the three terms: \(C\), \(C_K\), and \(C_T\).

- More instruments can be used to match higher-order expansion terms.
- Via the local volatility linkage between \(C_T\) and \(C_{KK}\), we can use 3 instruments to match 4 expansion terms, allowing us to go second order in strike.

The approach can readily accommodate three different maturities, different expansion points, and more expansion instruments.
Standardized strike and maturity spacing

- The portfolio weights depend on the strike-maturity layout.

- We define standardized measures for strike spacing and maturity spacing, respectively

  1. Standardized strike spacing:

     \[ d_j \equiv \frac{(K_j - K)}{K \sigma(K, T_o) \sqrt{T - T_o}}, \quad j = d, c, u, \]

     which approximates the number of standard deviations that the security price needs to move from \((T_o, K_j)\) to \((T, K)\).

  2. Standardized maturity spacing:

     \[ \alpha \equiv \frac{T_o - T_c}{T - T_o}, \quad (1) \]

     which measures the relative distance between the two maturities in the hedge triangle to the distance between the target option maturity and the reference hedge maturity \(T_o\).
The triangle portfolio weights

- The portfolio weights for the maturity-strike triangle are given as a function of the standardized strike and maturity spacing,

\[
\begin{bmatrix}
  w_d \\
  w_c \\
  w_u
\end{bmatrix}
= \begin{bmatrix}
  1 & 1 & 1 \\
  K_d & K_c & K_u \\
  d_d^2 & d_c^2 - \alpha & d_u^2
\end{bmatrix}^{-1}
\begin{bmatrix}
  1 \\
  K \\
  1
\end{bmatrix}.
\]

The portfolio weights are approximately static.

- When the strike spacing are symmetric with \( K_c = K \) and \( K_u - K = K - K_d \), the weights on the isosceles triangle are,

\[
\begin{align*}
  w_c &= \frac{d^2 - 1}{d^2 + \alpha}, \\
  w_d &= w_u = \frac{1}{2} (1 - w_c).
\end{align*}
\]

where \( d = d_u = |d_d| \).

- When \( T_c = T_o \) (a degenerate line), \( w_c = 1 - 1/d^2 \).

  - Carr and Wu’s static hedging with short-term options:

    - A quadrature approximation of the continuum with three strikes coincides with our degenerate line strategy with \( d = \sqrt{3} \), \( w_c = 2/3 \).
Center strike weight as a function of strike spacing

Ideally, choose $d > 1$ for stability.
From expansion errors to hedging errors

- Taylor expansion errors increase with the expansion distance.
- Strike distance can be chosen small, but maturity distance is likely large. Hence, we potentially have large expansion errors on maturity.
- **Hedging errors can be small even if expansion errors are large.**
  - Expansion errors in the target options partially cancel with expansion errors in the hedge portfolio.
  - The portfolio weights do not depend on expansion points.
  - When many strikes are available, one can choose the strike spacing judiciously to further increase the expansion error cancelation.
Monte Carlo analysis

- Simulate four model dynamics.

\[
\begin{align*}
\text{BS:} & \quad dS_t/S_t = \mu dt + \sigma dW_t, \\
\text{MJ:} & \quad dS_t/S_t = \mu dt + \sigma dW_t + \int_{\mathbb{R}_0} (e^x - 1) (\nu(dx, dt) - \lambda n(x) dx dt), \\
\text{HV:} & \quad dS_t/S_t = \mu dt + \sqrt{\nu_t} dW_t, \\
\text{HW:} & \quad dS_t/S_t = \mu dt + \sqrt{\nu_t} dW_t + \int_{\mathbb{R}_0} (\nu(dx, dt) - \nu_t \lambda_0 n(x) dx dt), \\
& \quad d\nu_t = \kappa (\theta - \nu_t) dt - \omega \sqrt{\nu_t} dZ_t, \quad \mathbb{E}[dZ_t dW_t] = \rho dt,
\end{align*}
\]

Parameters are set to averages of daily calibration results to SPX options.

- Perform hedging exercises of different target options with different maturity-strike combinations to learn

  - How the strike spacing choice affects the hedging performance under different model environments.
  - How the hedging performances compare with daily delta hedging.
Simulation procedures

- In each simulation, we generate a time series of daily underlying security prices according to an Euler approximation of the data generating process.

- The starting value for the stock price is normalized to $100, and the starting values of the instantaneous variance rates for the HV and HW models are also fixed to the average values from the daily calibration.

- At the start of each simulation, options are available at maturities of one, two, three, six, and 12 months, and that option strikes are centered around the normalized spot price of $100, and spaced at intervals of $1, $1.5, $2, $2.5, and $3 for the five maturities, respectively.

- To compute the portfolio weights, we estimate the local volatility by interpolating the implied volatility surface constructed from the finite number of option observations.

- We consider a hedging horizon of one month. The simulation starts on a Wednesday and ending on a Thursday 4 weeks later, for 21 week days.

- The hedging error at each date $t$, $e_t$, is defined as the difference between the value of the hedge portfolio and the value of the target call option.
Security price and volatility sample paths

BS

HV

HV volatility

MJ

HW

HW volatility

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Hedging performance under different strike spacing

BS

MJ

There are 20 more maturity-combinations...
Appropriate strike spacing choice can significantly reduce hedging error.
Hedging performance under different strike spacing

Appropriate strike spacing choice can significantly reduce hedging error.
Optimal strike spacing as a function of maturity spacing

Optimal strike spacing $d^*$ (that minimizes terminal RMSE) is related to the relative maturity spacing among the hedge options $\alpha = (T_o - T_c)/(T - T_o)$ and between hedge and target options $(T_o/T)$

$$d^* = a + b\alpha + c(T_o/T) + e,$$

<table>
<thead>
<tr>
<th>Model</th>
<th>$a$</th>
<th>$b$</th>
<th>$c$</th>
<th>$R^2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>BS</td>
<td>1.3843 (0.0292)</td>
<td>-0.6210 (0.0423)</td>
<td>0.3899 (0.0890)</td>
<td>0.8981</td>
</tr>
<tr>
<td>MJ</td>
<td>1.2383 (0.0168)</td>
<td>-0.5661 (0.0244)</td>
<td>0.6976 (0.0513)</td>
<td>0.9491</td>
</tr>
<tr>
<td>HV</td>
<td>1.0495 (0.0837)</td>
<td>-1.0712 (0.1212)</td>
<td>1.9006 (0.2549)</td>
<td>0.7430</td>
</tr>
<tr>
<td>HW</td>
<td>1.1082 (0.0886)</td>
<td>-0.9661 (0.1283)</td>
<td>1.2072 (0.2697)</td>
<td>0.6623</td>
</tr>
</tbody>
</table>

The higher $T_o$ relative the center strike $T_c$ and lower relative to target $T$, the narrower the strike spacing.

We can use these estimated relations to choose strike spacing in practice.
Hedging performance comparison

\[ T_h – \text{Average triangle maturity} \]

BS

MJ

HV

HW

The closer the target is to the triangle, the better the performance.

<table>
<thead>
<tr>
<th>T</th>
<th>BS</th>
<th>MJ</th>
<th>HV</th>
<th>HW</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>0.21</td>
<td>0.85</td>
<td>0.88</td>
<td>1.08</td>
</tr>
<tr>
<td>3</td>
<td>0.17</td>
<td>0.76</td>
<td>0.95</td>
<td>1.07</td>
</tr>
<tr>
<td>6</td>
<td>0.12</td>
<td>0.60</td>
<td>0.89</td>
<td>1.02</td>
</tr>
<tr>
<td>12</td>
<td>0.08</td>
<td>0.43</td>
<td>0.68</td>
<td>0.86</td>
</tr>
</tbody>
</table>

- BS: Triangles with \( T / T_h < 3 \) perform better.
- MJ: All triangles perform better.
- HV: All triangles perform better.
- HW: All triangles perform better.
Hedging error sample paths

Compare the best delta (top panels) with the worst triangle (bottom panels):

- BS
- MJ
- HV
- HW

- Delta hedge: Negative gamma → Large move leads to large loss.
- Triangle: Reasonably symmetric.
Root mean squared hedging error

Compare the best delta (dashed line) with the worst triangle (solid line):

- BS
- MJ
- HV
- HW

![Graphs showing RMSE over days for BS, MJ, HV, and HW models.]

- Delta hedge effectiveness depends crucially on dynamics. The performance is very good under BS, but deteriorates drastically in the presence of jumps/stochastic volatility. RMSE under HW is 9 times RMSE under BS.

- Triangle: Performance is stable across all model environments. RMSE are between 0.2-0.4 under all models.
A historical exercise on SPX options

- Daily data on SPX options from January 1996 to March 2009.
- Choose 158 starting dates with a set of options expiring in exactly 30 days.
- At each starting date, group options into 4 maturity groups: (i) 1 month (30) days, (ii) 2 months (59 or 66 days), (iii) 3-5 months (87-157 days), (iv) one year (276-402 days).
- Based on the 4 maturity groups, form 14 target-hedge portfolio maturity combinations: 4 with $T_c < T_o < T$, 4 with $T_o < T_c < T$, and 6 with $T_c = T_o < T$.
- Choose the target option strike $K$ closest to the spot level.
- Choose optimal strike spacing based on the regression results from simulation. Map the optimal strike spacing to the closest available strikes.
- Construct the local volatility surface from the observed option implied volatilities. Compute weights for each portfolio.
- Track the hedging error over 30 days as in the simulation.
- Perform delta hedging for comparison.
The hedging errors are slightly larger than the HV&HW case due to constraints in strike availability and/or more complicated dynamics.

Daily delta-hedge performance:
- RMSE on 2, 4, and 12-month options is 0.63, 0.63, 0.66.
- 11 of the 14 triangles perform better, even when target maturity is 6 times of hedge maturity.
Hedging error sample paths on SPX options

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Concluding remarks

- Existing hedging practices are mostly based on risk-exposures.
- The issue: It is hard to know exactly what the risk exposures are.
  - Different model assumptions can all match current market prices, but can imply quite different hedging ratios.
- We propose a simple, robust hedging strategy that does not depend on risk exposures (model assumptions), but is based purely on affinities of contracts in terms of strike and maturity.
  - It does not ask for a model, nor does it ask for a continuum of options.
- The strategy relies on maturity-strike triangles that can be constructed flexibly to balance contract availability, transaction cost, and hedging efficiency.
- Simulation exercise shows that a wide range of triangles can be formed under practical constraints to perform better than delta hedging with daily rebalancing.
- A historical run on SPX options confirm that most triangles outperform delta hedge.