Simple Robust Hedging with Nearby Contracts

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Archimedes-style hedging

*Give me a lever long enough and a fulcrum on which to place it, and I shall move the world.*
— Archimedes, Mathematician and inventor of ancient Greece, 287-212BC

- In hedging derivatives risk, many think like Archimedes, by making strong, idealistic assumptions on the security dynamics and trading environment.
  - Black-Merton-Scholes (1973) introduce the dynamic hedging concept, by assuming
    - The underlying security follows a one-factor diffusion process.
    - One can rebalance the hedge portfolio continuously.
  - Carr and Wu (2002) propose a static hedge on vanilla options, by assuming
    - The underlying security follows a one-factor Markovian process.
    - One can deploy an infinite number of short-term options across the whole continuum of strikes.

- Both results are ground breaking, but both rely on strong, Archimedes-style assumptions.
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In reality, the lever is not quite as long

- Transaction cost is a fact of life. Both continuous rebalancing and transacting on a continuum of options lead to financial ruin.
  - Discretization is a must.

- One does not know the exact dynamics of the underlying security: (i) how many risk sources and (ii) what the risk exposures are.
  - Riskfree hedge under model A is not riskfree under model B.
  - Neither hedge is riskfree in reality.

- Archimedes (or an Archimedes-style hedger) does not care. All he wants is to guarantee that he’s absolutely right, under his own terms.

- Practitioners do care, as they want to be approximately right (so that they do not lose their shirts), under all conceivable conditions.
Practitioners’ remedy

- Use BMS model to perform both delta and vega hedge.
  - Delta is balanced daily.
  - Vega and stress risk are managed opportunistically.

- Acknowledge that *this* vega is not *that* vega.
  - Vega at different strike and maturity ranges are different types of vegas.

- Example I: A portfolio with long $10 million vega at 5-year at-the-money and short $10 million vega at 4-year at-the-money is treated as relatively vega-risk free — vega is netted.

- Example II: A portfolio with long $10 million vega at 5-year at-the-money and short $10 million vega at 1-month 10-delta put is treated as having significant ($20 million) vega exposure — vega is added.

- From academic perspective, stochastic volatility can be generated from diffusion risk or jump risk, market risk or credit risk, short-term shock or long-term trend shift...
Our approach: Hedging with nearby contracts

- Academics often worry about the exact risk exposure calculation: Should the delta be calculated under a local vol model or a model with jump?
  - With the right exposure estimate, one can pick any contracts to neutralize the risk by solving a system of equations.

- Traders are less pedantic about the calculation, but intuitively realize that
  - Achieving vega (duration, etc) neutrality with nearby contracts is safer than using contracts that are far away (in maturity and/or strike).

- We do not attempt to eliminate risks completely under a hypothetical model.

- We devise a simple robust hedging strategy that limits loss under all possible scenarios, regardless of model assumptions.
  - We do not assume a model, nor do we calculate risk exposures.
  - We design the hedging portfolio based on affinities in contract characteristics (such as strike and maturity), not risk exposures (such as delta, vega).
We hedge a target vanilla call option at strike $K$ and expiry $T$, $C(K, T)$, with three nearby call option contracts:

- Using more than three contracts to hedge is not practical given transaction cost — Forget about a continuum.
- $K_d < K_c < K_u$, with $K \in (K_d, K_u)$ and ideally $K_c = K$ when available.
- No theoretical constraints on the maturity choice, but we apply some practical considerations:
  - Since often fewer maturities are available than strikes, we focus on two maturities instead of three, with $K_c$ at one maturity $T_c$ and $(K_d, K_u)$ at another maturity $T_o$ to form a maturity-strike triangle.
  - It would be nice to sandwich the target maturity $T \in (T_c, T_o)$, but since short-term options tend to be more liquid than long-term options, we might need to choose $(T_c, T_o) < T$. 

![Maturity-strike triangle diagram](https://example.com/triangle.png)
Assumptions

- We assume that there are a finite number of (at least 3) options for us to choose to form the hedge portfolio.

- We use these options (one is enough) to compute a local volatility,

\[ \sigma^2(K, T_o) = \frac{2C_T(K, T_o)}{C_{KK}(K, T_o)}. \]

- The concept of local volatility is originally developed by Dupire (94) under a one-factor diffusion setting.

- Positive local volatility exists under a much more general setting.

- We are not concerned with the dynamics, but rather try to obtain a stable estimate of the relation via interpolations and extrapolations:
  - Details ...
  - What data are sparse, super-smooth.
Deriving portfolio weights on the maturity-strike triangle

The portfolio weights are obtained via the following steps:

1. Taylor expand both target and hedge options around \((K, T_o)\) to first order in \(T\) and second-order in \(K\):

\[
\begin{align*}
C(K, T) &\approx C + C_T(T - T_o), \\
C(K_d, T_o) &\approx C + C_K(K_d - K) + \frac{1}{2} C_{KK}(K_d - K)^2, \\
C(K_u, T_o) &\approx C + C_K(K_u - K) + \frac{1}{2} C_{KK}(K_u - K)^2, \\
C(K_c, T_c) &\approx C + C_K(K_c - K) + C_T(K, T_o)(T_c - T_o) + \frac{1}{2} C_{KK}(K_c - K)^2.
\end{align*}
\]

2. Replace \(C_{KK}\) with \(C_T\) via the local volatility definition.

3. Choose the three weights \((w_d, w_c, w_u)\) to match coefficients on the three terms: \(C\), \(C_K\), and \(C_T\).

- More instruments can be used to match higher-order expansion terms.
- Via the local volatility linkage between \(C_T\) and \(C_{KK}\), we can use 3 instruments to match 4 expansion terms, allowing us to go second order in strike.

\(\Rightarrow\) By matching the coefficients of each term, we do not need to know the values or their derivatives \((C, C_K, C_T)\), which would be model dependent.
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The triangle portfolio weights

When the strike spacing are symmetric with $K_c = K$ and $K_u - K = K - K_d$, the weights on the isosceles triangle are,

$$w_c = \frac{d^2 - 1}{d^2 + \alpha}, \quad w_d = w_u = \frac{1}{2}(1 - w_c).$$

where

- $d = \frac{(|K_o - K|)}{K\sigma(K, T_o)\sqrt{T - T_o}}$ is a standardized strike spacing measure.
- $\alpha = \frac{T_o - T_c}{T - T_o}$ is a relative maturity spacing measure.
- The portfolio weights are approximately static.
Taylor expansion errors increase with the expansion distance.

Strike distance can be chosen small, but maturity distance is likely large. Hence, we potentially have large expansion errors on maturity.

**Hedging errors can be small even if expansion errors are large.**

- Expansion errors in the target options partially cancel with expansion errors in the hedge portfolio.
- The portfolio weights do not depend on expansion points.
- When many strikes are available, one can choose the strike spacing judiciously to further increase the expansion error cancelation.
Monte Carlo analysis

● Simulate four model dynamics.

BS: \[ \frac{dS_t}{S_t} = \mu dt + \sigma dW_t, \]

MJ: \[ \frac{dS_t}{S_t} = \mu dt + \sigma dW_t + \int_{\mathbb{R}^0} (e^x - 1) \left( \nu(dx, dt) - \lambda n(x)dxdt \right), \]

HV: \[ \frac{dS_t}{S_t} = \mu dt + \sqrt{v_t} dW_t, \]

HW: \[ \frac{dS_t}{S_t} = \mu dt + \sqrt{v_t} dW_t + \int_{\mathbb{R}^0} \left( \nu(dx, dt) - v_t \lambda_0 n(x)dxdt \right), \]

\[ dv_t = \kappa (\theta - v_t) dt - \omega \sqrt{v_t} dZ_t, \quad \mathbb{E}[dZ_t dW_t] = \rho dt, \]

● Parameters are set to averages of daily calibration results to SPX options.

● Maturities and strikes are set similar to that for SPX options.
  ● Maturities: 1,2,3,6,12 months.
  ● Strikes are spaced around spot at $1, $1.5, $2, $2.5, and $3, resp.

● Perform hedging exercises of different target options with different maturity-strike combinations to learn
  ● How does the strike spacing choice affect the hedging performance under different model environments?
  ● How does the hedging performance compare with daily delta hedging?
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Hedging performance under different strike spacing

BS

There are 20 more maturity-combinations for each model... and 3 other models...

- Choosing the appropriate strike spacing can further reduce hedging error.
- The following regression seems to describe the optimal strike choice well:
  \[ d^* = a + b\alpha + c(T_o/T) + e. \]
- More analytical study can be useful to understand the underlying mechanism of dependence...
Hedging performance comparison

The closer the target is to the triangle, the better the performance.

**Th – Average triangle maturity**
- BS
- MJ
- HV
- HW

Daily delta hedging with underlying futures

<table>
<thead>
<tr>
<th>T</th>
<th>BS</th>
<th>MJ</th>
<th>HV</th>
<th>HW</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>0.21</td>
<td>0.85</td>
<td>0.88</td>
<td>1.08</td>
</tr>
<tr>
<td>3</td>
<td>0.17</td>
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<td>6</td>
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<td>0.89</td>
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<tr>
<td>12</td>
<td>0.08</td>
<td>0.43</td>
<td>0.68</td>
<td>0.86</td>
</tr>
</tbody>
</table>

- BS: Triangles with \( T / T_h < 3 \) perform better.
- MJ, HV, HW: All 30 triangles perform better than delta hedging.
- **Not always nearby:** The target-hedge maturity ratio can be 12 times.
Hedging error sample paths

Compare the best delta (top panels) with the worst triangle (bottom panels):

- BS
- MJ
- HV
- HW

- Delta hedge: Negative gamma $\rightarrow$ Large move leads to large loss.
- Triangle: Reasonably symmetric.
Root mean squared hedging error

Compare the best delta (dashed line) with the worst triangle (solid line):

- Delta hedge effectiveness depends crucially on dynamics. The performance is very good under BS, but deteriorates drastically in the presence of jumps/stochastic volatility. RMSE under HW is 9 times RMSE under BS.

- Triangle: Performance is stable across all model environments. RMSE are between 0.2-0.4 under all models.
A historical exercise on SPX options

- Daily data on SPX options from January 1996 to March 2009.
- Choose 158 starting dates with a set of options expiring in exactly 30 days.
- At each starting date, group options into 4 maturity groups: (i) 1 month (30) days, (ii) 2 months (59 or 66 days), (iii) 3-5 months (87-157 days), (iv) one year (276-402 days).
- Based on the 4 maturity groups, form 14 target-hedge portfolio maturity combinations: 4 with $T_c < T_o < T$, 4 with $T_o < T_c < T$, and 6 with $T_c = T_o < T$.
- Choose the target option strike $K$ closest to the spot level.
- Choose optimal strike spacing based on the regression results from simulation. Map the optimal strike spacing to the closest available strikes.
- Construct the local volatility surface from the observed option implied volatilities. Compute weights for each portfolio.
- Track the hedging error over 30 days as in the simulation.
- Perform delta hedging for comparison.
Hedging performance comparison on SPX options

Triangle performance:

The hedging errors are slightly larger than the HV&HW case due to constraints in strike availability and/or more complicated dynamics.

Daily delta-hedge performance:

- RMSE on 2, 4, and 12-month options is 0.63, 0.63, 0.66.
- 11 of the 14 triangles perform better, even when target maturity is 6 times of hedge maturity.
Concluding remarks

- Existing hedging practices are mostly based on risk-exposures.
- The issue: It is hard to know exactly what the risk exposures are.
  - Different model assumptions can all match current market prices, but can imply quite different hedging ratios.
- We propose a simple, robust hedging strategy that does not depend on risk exposures, but only on affinities of contracts characteristics.
- We explore the idea using the example of hedging one option with three options that form a maturity-strike triangle. The results are promising.
  - Many triangles can be constructed flexibly to balance contract availability, transaction cost, and hedging efficiency.
  - Both simulation and a historical run on SPX options show that most triangles outperform delta hedge under realistic environments.
- A lot more can be explored on this new idea: (i) other markets, (ii) large portfolio, (iii) optimal contract placement, (iv) hedging error behavior and dependence on contract placements ...