Overview of standard option pricing models

The approach:
- Specify the full risk-neutral dynamics for the underlying security price.
- Take expectation of terminal payoffs according to the same dynamics.

The good: \textit{cross-sectional consistency}
- The dynamics specification provides a \textit{single yardstick} for valuing all options on the same underlying security.
- The valuations on different contracts are consistent with one another, in the sense that they are all derived from the same yardstick.
- Even if the yardstick is wrong, the valuations remain consistent with one another — They are just consistently wrong.

The bad: \textit{It is hard to get everything under one roof.}
- Pricing long-dated contracts requires unrealistically long projections.
- Short-term variations often look incompatible with long-run (stationarity) assumptions (Giglio & Kelly, 2017)
- Disturbance on one contract affects everything else.
A new pricing framework for short-term investors

- One can invest in a very long-dated contract for a very short time.
- A short-term investor does not care much about the *terminal* payoff.
  - All she worries about is the P&L on *her particular investment* over the *short investment horizon*.
- We develop a new framework that generates pricing implications for *short-term* investors interested in a *particular set of contracts*.
  - The short-term focus allows us to make near-term risk projections without worrying about their future variation and long-run dynamics.
  - The “particular-contract” focus allows us to make local inferences relevant to the particular investor, without generating (potentially unrealistic) implications on other contracts.
  - "*Aim small, miss small.*"
    - The narrow focus allows one to make more relevant, more accurate projections, leading to more accurate pricing implications.
    - Remove tensions between short-run v. long-run expectations, and across different contracts (e.g., short-dated v long-dated, OTM call v. OTM put).
P&L attribution of a short-term option investment

Imagine we have a position in a vanilla option. We start by performing a short-term (daily) P&L attribution analysis on this position.

- We represent the option via the BMS pricing equation, $B(t, S_t, I_t; K, T)$
  - $(K, T)$ capture the contract characteristics.
  - The value of the contract can vary with calendar time $t$, the underlying security price $S_t$, and the option’s BMS implied volatility $I_t$.
  - As long as the option price does not allow arbitrage against the underlying and cash, there existence of a positive $I_t$ to match the price.
  - This is common practice, but one can explore other representations...

- The investment P&L over the next instant can be decomposed as
  $$
  dB = \left[ B_t dt + B_S dS_t + B_I dI_t \right] + \left[ \frac{1}{2} B_{SS} (dS_t)^2 + \frac{1}{2} B_{II} (dI_t)^2 + B_{IS} (dS_t dI_t) \right] + \text{Jump},
  $$
  - The expansion can stop at second order for diffusive moves.
  - It is harder to attribute jump risk via expansion — more used for scenario analysis/stress tests.
Take risk-neutral expectation on the investment P&L, and divide by the investment horizon $dt$, we can attribute the expected investment return as:

$$E_t \left[ \frac{dB}{dt} \right] = B_t + B_l I_t \mu_t + \frac{1}{2} B_{SS} S_t^2 \sigma_t^2 + \frac{1}{2} B_{II} I_t^2 \omega_t^2 + B_{IS} I_t S_t \gamma_t$$  \hspace{1cm} (1)

$\mu_t, \sigma_t^2, \omega_t^2, \gamma_t$ denote the time-$t$ conditional mean, variance, and covariance:

$$\mu_t \equiv E_t \left[ \frac{dl_t}{l_t} \right] / dt, \quad \sigma_t^2 \equiv E_t \left[ \left( \frac{dS_t}{S_t} \right)^2 \right] / dt,$$

$$\omega_t^2 \equiv E_t \left[ \left( \frac{dl_t}{l_t} \right)^2 \right] / dt, \quad \gamma_t \equiv E_t \left[ \left( \frac{dS_t}{S_t}, \frac{dl_t}{l_t} \right) \right] / dt.$$

With zero financing assumption, the risk-neutral expected return on the security is 0, so is the expected return on the option:

$$0 = B_t + B_l I_t \mu_t + \frac{1}{2} B_{SS} S_t^2 \sigma_t^2 + \frac{1}{2} B_{II} I_t^2 \omega_t^2 + B_{IS} I_t S_t \gamma_t.$$

This can be regarded as a pricing equation: The value of the option must satisfy this equality to exclude dynamic arbitrage.
P&L attribution relative to pricing expectation

It is more informative to represent the P&L attribution in terms of surprises against pricing expectations:

\[
\begin{align*}
    dB &= B_SS_t \frac{dS_t}{S_t} + B_1 I_t \left( \frac{dl_t}{l_t} - \mu_t dt \right) \\
    &+ \frac{1}{2} B_{SS} S_t^2 \left( \left( \frac{dS_t}{S_t} \right)^2 - \sigma_t^2 dt \right) \\
    &+ \frac{1}{2} B_{II} I_t^2 \left( \left( \frac{dl_t}{l_t} \right)^2 - \omega_t^2 dt \right) \\
    &+ B_{IS} S_t I_t \left( \frac{dS_t}{S_t} \frac{dl_t}{l_t} \right) - \gamma_t dt
\end{align*}
\]

\( (B_S, B_I, B_{SS}, B_{II}, B_{IS}) \) are risk exposures/loadings, surprises are risk.

- Subjective–pricing expectation deviation defines alpha (or risk premium).
- Deviation of realization from expectation defines risk.
- Price move \( (dS_t) \) is the main risk source — BMS delta hedge can remove over 90% of variation (Figlewski, 89).
- Vega hedge using ATM contracts of the same maturity can remove most vol risk. — Implied vols at same maturity move mostly together.
- What to hedge/bet is a risk-return trade-off problem.
Moment-based implied volatility valuation

- Start with the pricing relation,
  \[ 0 = B_t + B_l l_t \mu_t + \frac{1}{2} B_{SS} S_t^2 \sigma_t^2 + \frac{1}{2} B_{II}^2 \omega_t^2 + B_{IS} l_t S_t \gamma_t, \]

- Represent the risk exposures in terms of dollar gamma:
  \[
  B_t = -\frac{1}{2} l_t^2 B_{SS} S_t^2, \quad B_l l_t = l_t^2 \tau B_{SS} S_t^2, \quad B_{IS} l_t S_t = (k + \frac{1}{2} l_t^2 \tau) B_{SS} S_t^2, \quad B_{II} l_t^2 = (k + \frac{1}{2} l_t^2 \tau) (k - \frac{1}{2} l_t^2 \tau) B_{SS} S_t^2. \]

- \( k \equiv \ln K / F_t \) — the relative strike of the option
- \( z_\pm \equiv (k \pm \frac{1}{2} l_t^2 \tau) \) — convexity-adjusted moneyness

- Divide the pricing relation by dollar gamma,
  \[
  l_t^2 = \left[ 2 \tau \mu_t l_t^2 + \sigma_t^2 \right] + \left[ 2 \gamma_t z_+ + \omega_t^2 z_+ z_- \right].
  \]

- This can be regarded as a local no-arbitrage implied volatility valuation on a particular option contract.

- The no-arbitrage condition determines the fair implied volatility level \( l_t \) of the option contract in terms of the first and second risk-neutral moments of the joint move \( \left( \frac{dS_t}{S_t}, \frac{dl_t}{l_t} \right) \): \( (\mu_t, \sigma_t^2, \omega_t^2, \gamma_t). \)
Moment-based implied volatility valuation

\[ I_t^2 = [2\tau \mu_t I_t^2 + \sigma_t^2] + [2\gamma_t z_+ + \omega_t^2 z_+ z_-] \]

- The no-arbitrage condition determines the time-\( t \) fair implied volatility level \( I_t \) of the option contract in terms of the time-\( t \) conditional first and second risk-neutral moments of the joint move \((dS_t S_t, \frac{dl}{l_t})\): \((\mu_t, \sigma_t^2, \omega_t^2, \gamma_t)\).

- It ties the current value of the contract to the current moment conditions, or risk profile of the contract, with no reference to
  - how the risk profile varies over time,
  - how it behaves in the long run
  - how the risk profile is estimated/derived/modelled

- The valuation is not linked to any other option contracts, unless we specifically want to make a linkage, e.g., via common factor structures on the risk profiles of these contracts.
The above pricing equation links the implied volatility level of one option contract to its own risk profile.

Joint valuation of the implied volatility surface (or a particular region of it) requires additional commonality assumptions on their risk profiles.

- A potential place for future work to understand the form of commonality in risk profiles ...
- Assuming *locally parallel shifts* leads to *locally smooth surfaces*

Joint valuation across names requires common factor loading structures across names, e.g., BARRA.

PCA often identifies 3 major sources of variation on the surface:

- The overall volatility level
- Term structure variation (short v. long-dated contracts)
- Implied volatility smile/skew variation along moneyness (OTM put v. straddle v. OTM call)

We perform separate analysis of the two dimensions.
At-the-money contracts

- We define "at-the-money" as contracts with \( k = -\frac{1}{2} \int_t^\tau \)
- This is the only contract whose implied volatility level only depends on its expected move \((\mu_t)\), but not its variance/covariance:

\[
A_t^2 = 2\tau \mu_t A_t^2 + \sigma_t^2.
\]

Applications:

- **1 contract**: Infer risk-neutral drift from the slope against instantaneous variance:
  \[
  \mu_t = \frac{A_t^2 - \sigma_t^2}{2A_t^2\tau}.
  \]
- **2 contracts**: Infer *locally common* drift from the slope of nearby at-the-money contracts:
  \[
  \mu_t = \frac{A_t^2(\tau_2) - A_t^2(\tau_1)}{2(A_t^2(\tau_2)\tau_2 - A_t^2(\tau_1)\tau_1)}.
  \]

Locally constant drift leads to locally linear term structure — Drift estimates can be tied to local linear (nonparametric) regression fitting of the term structure.
The at-the-money implied variance term structure

\[ A_t^2 = 2\tau \mu_t A_t^2 + \sigma_t^2. \]

Build the at-the-money *term structure function* over the whole maturity range or a particular maturity segment based on commonality assumptions on their drifts.

- Example: All ATM implied variance share the same mean-reversion and proportional variance structure *at time* \( t \):

\[ dA_t^2(\tau) = \bar{\kappa}_t (\bar{\theta}_t - A_t^2(\tau))dt + 2\omega_t A_t^2(\tau) dZ_t \]

Implication on the implied variance term structure function:

\[ A_t^2(\tau) = \frac{1 - e^{-\kappa_t \tau}}{\kappa_t \tau} (\sigma_t^2 - \theta_t) + \theta_t. \]

- Assumptions similar to standard stochastic volatility models lead to similar term structure implications, but still with more freedom:
- If the common dynamics assumptions/estimates change tomorrow, so will the term structure.
- Different from standard option pricing models, the current term structure pricing does not depend on its future risk profile variation.
The implied volatility smile at a single maturity

- To highlight the implied volatility smile at a certain maturity, we can vega hedge the option with the at-the-money contract of the same maturity, assuming they strongly co-move.
- Take the ATM implied variance $A_t^2$ as given and focus on the implied variance deviation of other contracts from the ATM variance level.
- Assume proportional drift at the same maturity: $\mu_t l_t^2 = \mu_t A_t^2$.
- Plug in the at-the-money implied variance to highlight the “implied volatility smile” effect at the single maturity,
  \[ l_t^2 - A_t^2 = 2\gamma_t z_+ + \omega_t^2 z_+ z_- , \]  
  (2)
  
  - $z_\pm = k \pm \frac{1}{2} l_t^2 \tau$ denote the two convexity-adjusted moneyness measures.
  - The smile slope is determined by the covariance rate $\gamma_t$ of the contract.
  - The smile curvature is determined by the variance rate $\omega_t^2$.

Application:
Assuming common proportional implied volatility movements within a particular moneyness range, we can identify the common moment conditions $(\omega_t^2, \gamma_t)$ by regressing $l_t^2 - A_t^2$ against $(2z_+, z_+ z_-)$.
A local quadratic implied variance smile

- Represent the smile in two moneyness measures $z_{\pm}$ is not that convenient.
- We can factor out the convexity terms and construct the implied variance smile as a function of the relative strike $k$,

$$I^2_t - A^2_t = 2\gamma_t k + \omega^2_t k^2$$

(3)

with $A^2_t = A^2_t + \gamma_t I^2_t \tau - \frac{1}{4} \omega^2_t I^4_t \tau^2$.

- If $A^2_t$ varies little with $k$, we can treat $A^2_t$ as the at-the-money forward implied variance (at $k = 0$), and treat (3) as a quadratic equation in $k$.
- Otherwise, (3) can be treated at an iterative quadratic equation.
- Assuming *locally common* risk profile $(\gamma_t, \omega^2_t)$, we can represent the implied volatility function as a *local quadratic* function in $k$.

- This allows us to link local quadratic smoothing of the implied variance smile to the locally smoothed variation of risk profiles.
Level/skew variation and polynomial smile

Smile shape depends on commonality assumptions on the risk profile.

- If we decompose the smile variation into variations in level ($\mathcal{A}_t$) and the skew slope ($\mathcal{L}_t$),

$$\ln I_t(k) = \ln \mathcal{A}_t + \mathcal{L}_t k,$$

$$\mathcal{L}_t = \frac{\ln I(k)/\mathcal{A}_t}{k}.$$ (4)

- The change in each implied volatility series for a fixed-strike contract can be represented as

$$d \ln I_t(k) = d \ln \mathcal{A}_t + k d \mathcal{L}_t - \mathcal{L}_t d \ln S_t,$$ (5)

with the last term due to sliding along the smile.

- We can approximately represent the smile as a 4th-order polynomial,

$$I^2_t(k) = \mathcal{A}_t^2 + C_{AA}k + \tilde{C}_{AA}k^2 + 2C_{AL}k^3 + C_{LL}k^4$$

with higher order terms driven by skew slope ($\mathcal{L}$) variation $C_{LL}$ and covariation with ATM vol $C_{AA}$.

- Future research: The right formulation depends on the commonality assumptions of the risk profiles...
How to extrapolate observed smile to a very long maturity?

**Different starting point leads to different extrapolations**

- Short-dated options are actively traded on the market. The implied volatility smiles are readily observable.
- Brokers are often asked to provide quotes on very long-dated options not traded on the exchange.
- The approach based on standard “fundamental” option pricing models
  - Calibrate a standard stochastic vol model to observed option quotes (say up to 2 years), price long-dated options (say 10 years) with the model parameter estimates.
  - The at-the-money vol level will flatten out to the long-run mean (roughly matching the 2-year ATM vol level)
  - The implied volatility smile/skew will disappear due to central limit theorem (and the fact that volatility converges to its long-run mean).
- Our pricing approach:
  - If the at-the-money vol is flat extrapolated from 2 to 10 years, the 2-year and 10-year at-the-money implied variance will vary by the same amount — same \( (\gamma_t, \omega_t^2) \).
  - The smile/skew shape must be extrapolated from 2 to 10 years as well!
Concluding remarks

Traditional pricing models tend to go *bottom-up* by specifying the dynamics of some basis reference, from which one derives the value for all contracts over all horizons.

We strive to build a more practical, *top-down* pricing approach by starting with the *particular contract* one is interested investing in and by considering its risk profile/exposure *one day at a time*.

- It sounds less ambitious, generates narrower implications, but the implications it generates tend to be more relevant and more accurate.
- For option contracts on the same underlying, one can build smoothed implied volatility surfaces (or a particular segment of the surface) via commonality assumptions, such as locally parallel movements.
- Across names, one can build up the risk structure via cross-name commonality assumptions, such as industry and other firm characteristics (e.g., BARRA).
- Future work is needed to explore its implications and applications...