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The variation of the interest-rate term structure is fully determined by the risk-neutral dynamics of the interest rates.

Reverse engineering: One often uses the term-structure variation to identify the risk-neutral interest-rate dynamics.

Given the identified dynamics, one can in principle price any interest-rate related securities (such as caps, floors, swaptions...).

Problems:

- Large portions of interest-rate option variation cannot be explained by the interest-rate term structure variation. –Collin-Dufresne and Goldstein (2002), Heidari and Wu (2003), Joslin (2006)
- Dynamic term structure models (DTSM) fitted to the interest-rate term structure do not price interest-rate options well. Dai and Singleton (2003), Li and Zhao (2006), and Heidari and Wu (2008)

Bottom line: Volatility movements are not that sensitive to the term structure variation, even if they are theoretically spanned by the term structure.
Lack of linkage between term structure and volatility

How to handle the lack of linkage between the term structure and volatility?

- Current practice (HJM type): Ignore each other.
  - Use dynamic term structure models to fit the term structure only. Volatility specifications do not matter much — simplify amap.
  - Option pricing takes the existing term structure as given (no term structure modeling) and only models volatility dynamics.

  - The $n$ factors have small impacts on the term structure (treated as transient error), but large impacts on option implied volatility.


- This paper explores the linearity-generating (LG) framework of Gabaix (2007): Bond prices are linear (instead of exponential affine) in factors.
  - Stochastic volatilities in the linear factors do not affect bond price, but affect option pricing.
  - Sequential identification: Identify the factor transition matrix from the term structure; and volatility dynamics from options.
A one-factor example: a partial specification

- Start with the instantaneous interest rate, \( r_t = \theta_r + x_t \), with the following *risk-neutral dynamics for the short-rate gap* \( x_t \):
  \[
  dx_t = -x_t (\kappa - x_t) \, dt + dn_t.
  \]

- Time-varying mean-reversion speed \((\kappa - x_t)\): high when the rate level is low, and low when the rate level is high.
- \( n_t \) denotes the (unspecified) martingale component—It could be jump, diffusion, or nothing (deterministic).

- The zero-coupon bond value is affine (*not exponential affine*) in \( x_t \):
  \[
  P(t, T) = \mathbb{E}_t^Q \left[ e^{-\int_t^T r_s \, ds} \right] = e^{-\theta_r \tau} \left( 1 - \frac{1-e^{-\kappa \tau}}{\kappa} x_t \right), \quad \tau = T - t.
  \]

- Bond pricing does not depend on the specification of the martingale component \( n_t \).
- Full specification is only needed for pricing options, thus a true separation of term structure modeling and option pricing.
A full specification with an alternative representation

- Start with the state price deflator \( M_t \) such that bond pricing is given by
  \[ P(t, T) = \mathbb{E}_t[M_T / M_t]. \]

- Alternative representation wrt a positive martingale \( Z_t \):
  \[
  M_t = \nu^\top Y_t, \quad \nu = [1, 1]^\top \\
  Y_t = e^{-A t} \left( \begin{array}{c}
  \alpha_0 \\
  \alpha + \beta Z_t
  \end{array} \right), \quad A = \begin{pmatrix} \theta_r & 0 \\
  0 & \kappa + \theta_r \end{pmatrix},
  \]

  \( M_t > 0 \) for all \( t \) dictates \( \beta \geq 0 \) and \( \alpha_0 + \alpha \geq 0 \).

  Normalize \( Z_0 = 1 \) and \( \alpha_0 = 1 - \alpha - \beta \) so that \( M_0 = 1 \).

- The vector \( Y_t \) is linked to the original factor \( x_t \) by
  \[
  F_t = \left( \begin{array}{c}
  1 - x_t / \kappa \\
  x_t / \kappa
  \end{array} \right) = \frac{Y_t}{M_t} \quad \Rightarrow \quad x_t = \frac{\kappa e^{-\kappa t} (\alpha + \beta Z_t)}{\alpha_0 + e^{-\kappa t} (\alpha + \beta Z_t)}.
  \]

- The same bond pricing result:
  \[
  P(t, T) = \mathbb{E}_t \left[ \frac{M_T}{M_t} \right] = \frac{1}{M_t} \mathbb{E}_t \left[ \nu^\top Y_T \right] = \\
  \nu^\top \frac{e^{-A T} Y_t}{M_t} = \nu^\top e^{-A T} F_t = e^{-\theta_r T} \left( 1 - \frac{1 - e^{-\kappa T}}{\kappa} \right) x_t.
  \]
Fully-specified factor dynamics

- Assume $Z_t$ follows a GBM, $dZ_t/Z_t = \sigma dW_t$.

- The risk-neutral factor dynamics for $x_t$,
  
  $$dx_t = -x_t(\kappa - x_t)dt + \frac{(x_{\text{max}} - x_t)(x_t - x_{\text{min}})}{x_{\text{max}} - x_{\text{min}}} \sigma dW_t,$$

  The volatility of $x_t$ is parabolic and becomes zero at the two boundaries, $[x_{\text{min}}, x_{\text{max}}]$, with $x_{\text{max}} = \kappa$, and $x_{\text{min}} = \kappa \frac{\alpha e^{-\kappa t}}{\alpha_0 + \alpha e^{-\kappa t}}$.

- Setting $\alpha = -\frac{\theta r}{\kappa} (1 - \beta)$ leads to $x_{\text{min}} = -\theta r$ and $r_{\text{min}} = 0$.

- The coefficient $\beta$ determines the market price of risk:
  
  $$\frac{dM_t}{M_t} = -r_t dt - \gamma_t \sigma dW_t,$$

  with $\gamma_t = -\frac{e^{-\kappa t} \beta Z_t}{\alpha_0 + e^{-\kappa t} (\alpha + \beta Z_t)}$.

- The instantaneous risk premium on $x_t$ is given by
  
  $$\mu_t = -(\kappa - x_t)\gamma_t^2 \sigma^2.$$  

  as $dn_t$ can also be written as $-(\kappa - x_t)\gamma_t \sigma dW_t$. 
It is convenient to price options with the $M_t = \nu^\top Y_t$ representation.

State-price deflated bond prices are linear in $Z_t$:
$$P(t, T) = \mathbb{E}_t \left[ \frac{M_T}{M_t} \right] = \frac{1}{M_t} e^{-\theta_r T} \left( 1 + e^{-\kappa T} \beta (Z_t - 1) \right).$$

So are the prices of deflated bond portfolios (or coupon bonds):
$$P_T = \sum_s P(T, s) = \frac{1}{M_T} \sum e^{-\theta_r s} (1 + e^{-\kappa s} \beta (Z_T - 1)).$$

Options on bond portfolios can be valued as:
$$O_t = \mathbb{E}_t \left[ \frac{M_T}{M_t} \left( DP_T - K \right)^+ \right] = \mathbb{E}_t \left[ (F_t + G_t Z_T)^+ \right].$$

Many specifications on $Z_t$ generate tractable pricing for both caps/floors and swaptions.

If $dZ_t / Z_t = \sigma dW$, the Black-Scholes call formula applies:
$$O_t = BSC \left( G_t Z_t, -F_t, \sigma \sqrt{T} \right).$$

Caveat: Option pricing remains difficult when $Z_t$ is multi-dimensional.
Multi-factor generalization

The $M_t = \nu^\top Y_t$ representation can be readily generated to multiple factors: $M_t = \nu^\top Y_t$, $Y_t = e^{-At} \left( \{\alpha_j + \beta_j Z_t\}_{j=1}^m \right)$, $F_t = Y_t/M_t$.

with an $(m + n)$ factor structure extension:

- The transition matrix $A \in \mathbb{R}^{(m+1) \times (m+1)}$ and the loading vector $\nu \in \mathbb{R}^{(m+1)}$ determine the $m$-dimensional interest rate structure.

- The innovation $Z_t$ can have an $n$-dimensional stochastic volatility structure, the specification of which is independent of the term structure.
Interest-rate transition dynamics and bond pricing

- Bond price: $P(t, T) = \mathbb{E}_t \left[ \frac{M_T}{M_t} \right] = \nu^\top e^{-A_T F_t}$.

- If the transition matrix $A$ has distinct real eigenvalues, one can diagonalize the transition matrix $A_d$ while setting $\nu_d = (1, \cdots, 1)^\top$ without losing any generality.

  - $A = \theta_r + \langle 0, \kappa_1, \cdots, \kappa_m \rangle$ with $0 < \kappa_1 < \cdots < \kappa_m$.
  - $F_t = (1 - \sum_{i=1}^m \frac{x_i}{\kappa_i}, \frac{x_1}{\kappa_1}, \cdots, \frac{x_m}{\kappa_m})^\top$.

- Bond price: $P(t, T) = e^{-\theta_r T} \left( 1 - \sum_{i=1}^m (1 - e^{-\kappa_i T}) \frac{x_t^i}{\kappa_i} \right)$.

- The short rate: $r_t = \theta_r + \sum_{i=1}^m x_t^i$.

- This “diagonalization” is not possible in exponential-affine models:
  - One can either diagonalize the transition matrix or the covariance matrix, but not both.
  - Here, the covariance matrix does not matter for bond pricing.

- Parsimonious, dimension-invariant specification:
  \[ \kappa_j = \kappa_r s^{m-j}, s > 1, \text{ power law scaling for frequency distribution} \]
Under the diagonalized factor structure, the spot rates are given by

\[ y(x_t, \tau) = -\frac{\ln P(x_t, \tau)}{\tau} = \theta_r - \frac{1}{\tau} \ln \left( 1 - \sum_i \frac{(1 - e^{-\kappa_i \tau})}{\kappa_i} x_t^i \right). \]

The instantaneous factor loadings at \( x_t = 0 \) are:

\[ \frac{\partial y(x_t, \tau)}{\partial x_t^j} \bigg|_{x=0} = \frac{(1 - e^{-\kappa_j \tau})}{\kappa_j \tau}, \]

the same as an exponential-affine model, but only at \( x_t = 0 \).

The loadings of all factors start at one, with the decay speed for each factor \( j \) determined by \( \kappa_j \).
Rotate the factor structure for better identification

- If we rotate the $\kappa$ matrix from diagonal to cascading (Calvet, Fisher, Wu),

\[
\begin{align*}
\text{Diagonal} & \quad \Rightarrow \quad \text{Cascade} \\
\kappa &= \begin{pmatrix} \kappa_r & 0 & 0 \\ 0 & \kappa_r s & 0 \\ 0 & 0 & 0 \kappa_r s^2 \end{pmatrix} \Rightarrow \kappa &= \begin{pmatrix} \kappa_r s^2 & -\kappa_r s^2 & 0 \\ 0 & \kappa_r s & -\kappa_r s \\ 0 & 0 & \kappa_r \end{pmatrix}
\end{align*}
\]

Bond pricing remains the same form,

\[
P(x_t, \tau) = e^{-\theta \tau}(1 - c^\top(I_m - e^{-\kappa t})x_t),
\]

with $c^\top = b^\top(\kappa)^{-1}$, $b = [1, 0, 0, ...]^\top$.

- but factor identification becomes easier as the factor loadings become more separated:
Statistical dynamics

- Bond pricing only needs the long-run mean $\theta_r$, the transition matrix $\kappa$, and the factor levels $x_t = \{x_t^j\}_{j=1}^m$.

- Model estimation needs the full statistical dynamics of $x_t$.

- Assume for now $dZ_t/Z_t = \sigma dW_t$, the statistical dynamics for $x_t$ becomes,

$$
\begin{align*}
\text{Risk-neutral drift:} & \quad dX_t = -(\kappa - b^T x_t)x_t dt \\
\text{Risk premium:} & \quad -(g_t - x_t(c^T g_t))(c^T g_t)\sigma^2 dt \\
\text{Volatility:} & \quad + (g_t - x_t(c^T g_t))\sigma dW_t \\
\text{with } g_t & = \frac{\beta Z_t}{\alpha_0 + c^T(\alpha + \beta Z_t)}.
\end{align*}
$$

- One can verify that the volatility (and risk premium) is zero at the two boundaries: $Z = 0$ and $Z = \infty$.

- More assumptions are needed to determine $g_t$ for the statistical dynamics.

- Bond pricing is independent of these assumptions.
Estimation

- Estimate parameters \((\theta_r, \kappa, s)\) and extract states \(\{x_t^j\}_{i=1}^m\) using 15 LIBOR and swap rate series.
  - LIBOR at 1, 2, 3, 6, 9, 12 months;
    swap at 2, 3, 4, 5, 7, 10, 15, 20, 30 years.
  - 10 years of weekly-sampled data from August 19, 1998 to August 20, 2008, 523 weeks.

- Regard \(X_t\) as hidden state and LIBOR/swap rates as noisy observations.
  - Euler approximate the assumed statistical dynamics for state propagation.
  - Given parameters, apply nonlinear filtering to extract the states \(X_t\).
  - Define the likelihood on the forecasting error of LIBOR/swap rates.
  - Maximize the likelihood to estimate the model parameters.
## Pricing errors on LIBOR/swap rates with 3 factors

<table>
<thead>
<tr>
<th>Mat</th>
<th>Exponential-affine</th>
<th>Linearity-generating</th>
</tr>
</thead>
<tbody>
<tr>
<td>Mean</td>
<td>Rmse</td>
<td>Max</td>
</tr>
<tr>
<td>1 m</td>
<td>-1.47</td>
<td>8.47</td>
</tr>
<tr>
<td>3 m</td>
<td>1.86</td>
<td>5.43</td>
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<tr>
<td>6 m</td>
<td>1.37</td>
<td>7.48</td>
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<tr>
<td>9 m</td>
<td>-1.08</td>
<td>6.94</td>
</tr>
<tr>
<td>12 m</td>
<td>-2.66</td>
<td>6.65</td>
</tr>
<tr>
<td>2 y</td>
<td>0.84</td>
<td>6.84</td>
</tr>
<tr>
<td>3 y</td>
<td>1.41</td>
<td>7.12</td>
</tr>
<tr>
<td>4 y</td>
<td>0.61</td>
<td>6.15</td>
</tr>
<tr>
<td>5 y</td>
<td>-0.11</td>
<td>5.23</td>
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<tr>
<td>10 y</td>
<td>-1.95</td>
<td>5.37</td>
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<tr>
<td>15 y</td>
<td>1.34</td>
<td>4.07</td>
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<tr>
<td>20 y</td>
<td>2.10</td>
<td>5.23</td>
</tr>
<tr>
<td>30 y</td>
<td>-1.37</td>
<td>8.41</td>
</tr>
<tr>
<td>Avg</td>
<td>-0.01</td>
<td>6.17</td>
</tr>
</tbody>
</table>

A 3-factor LG model performs similarly to a 3-factor exponential-affine model.
### Pricing errors on LIBOR/swap with high-dim LG models

|   | Mat | Mean | Rmse  | Max   | Auto | VR   | Mean | Rmse  | Max   | Auto | VR   |
|---|-----|------|-------|-------|------|------|------|-------|-------|------|------|------|-------|-------|-------|------|-------|------|-------|
| 1 m | -1.68 | 8.65 | 49.39 | 0.87  | 99.78 | 0.07 | 0.63 | 5.77  | 0.49  | 100.00 | 0.07 | 0.63 | 5.77  | 0.49  | 100.00 |
| 3 m | 1.86  | 5.35 | 41.95 | 0.83  | 99.93 | -0.08| 2.02 | 19.59 | 0.61  | 99.99  | -0.08| 2.02 | 19.59 | 0.61  | 99.99  |
| 6 m | 1.62  | 7.51 | 24.32 | 0.94  | 99.84 | 0.12 | 1.16 | 8.65  | 0.69  | 100.00 | 0.12| 1.16 | 8.65  | 0.69  | 100.00 |
| 9 m | -0.71 | 7.10 | 31.00 | 0.90  | 99.85 | 0.24 | 0.99 | 4.38  | 0.77  | 100.00 | 0.24| 0.99 | 4.38  | 0.77  | 100.00 |
| 12 m| -2.29 | 6.89 | 31.38 | 0.79  | 99.87 | -0.36| 1.32 | 5.22  | 0.11  | 99.99  | -0.36| 1.32 | 5.22  | 0.11  | 99.99  |
| 2 y | 0.66  | 7.03 | 21.50 | 0.81  | 99.80 | 0.09 | 1.20 | 4.52  | -0.00 | 99.99  | 0.09| 1.20 | 4.52  | -0.00 | 99.99  |
| 3 y | 0.69  | 7.14 | 31.13 | 0.87  | 99.73 | 0.02 | 0.73 | 3.49  | 0.33  | 100.00 | 0.02| 0.73 | 3.49  | 0.33  | 100.00 |
| 4 y | -0.32 | 6.28 | 29.65 | 0.90  | 99.75 | -0.10| 0.84 | 8.26  | 0.12  | 100.00 | -0.10| 0.84 | 8.26  | 0.12  | 100.00 |
| 5 y | -0.96 | 5.43 | 23.66 | 0.90  | 99.79 | -0.03| 0.76 | 4.74  | 0.25  | 100.00 | -0.03| 0.76 | 4.74  | 0.25  | 100.00 |
| 10 y| -1.12 | 5.03 | 19.03 | 0.94  | 99.71 | -0.14| 0.77 | 5.48  | 0.25  | 99.99  | -0.14| 0.77 | 5.48  | 0.25  | 99.99  |
| 15 y| 2.88  | 4.94 | 15.52 | 0.90  | 99.76 | 0.03 | 0.68 | 4.37  | 0.42  | 99.99  | 0.03| 0.68 | 4.37  | 0.42  | 99.99  |
| 20 y| 3.19  | 5.86 | 17.21 | 0.90  | 99.60 | -0.04| 0.90 | 9.00  | 0.32  | 99.99  | -0.04| 0.90 | 9.00  | 0.32  | 99.99  |
| 30 y| -2.35 | 8.45 | 23.88 | 0.92  | 98.81 | 0.09 | 0.78 | 6.76  | 0.15  | 99.99  | 0.09| 0.78 | 6.76  | 0.15  | 99.99  |
| Avg | 0.00  | 6.33 | 27.34 | 0.88  | 99.73 | -0.00| 1.03 | 7.42  | 0.36  | 99.99  | -0.00| 1.03 | 7.42  | 0.36  | 99.99  |

A high-dimensional LG model generates negligible pricing errors, and can be used as the basis curve for pricing options.
### Interest-rate factor dynamics

<table>
<thead>
<tr>
<th>$m$</th>
<th>$\theta_r$</th>
<th>$\kappa_r$</th>
<th>$s$</th>
</tr>
</thead>
<tbody>
<tr>
<td>3</td>
<td>0.0612</td>
<td>0.0691</td>
<td>5.3458</td>
</tr>
<tr>
<td>5</td>
<td>0.0527</td>
<td>0.0659</td>
<td>2.6262</td>
</tr>
<tr>
<td>7</td>
<td>0.0491</td>
<td>0.0807</td>
<td>2.3164</td>
</tr>
<tr>
<td>9</td>
<td>0.0481</td>
<td>0.0834</td>
<td>2.1583</td>
</tr>
<tr>
<td>11</td>
<td>0.0471</td>
<td>0.1008</td>
<td>1.7780</td>
</tr>
<tr>
<td>13</td>
<td>0.0479</td>
<td>0.1160</td>
<td>1.6719</td>
</tr>
<tr>
<td>15</td>
<td>0.0567</td>
<td>0.0838</td>
<td>1.6205</td>
</tr>
</tbody>
</table>

- With the dimension-invariant specification, all parameters are estimated with high statistical significance.

- With more factors, the frequency components are distributed more closely with each other ($s$ becomes smaller).
Pricing interest-rate options

- Option pricing: \( O_t = \mathbb{E}_t \left[ \frac{M_T}{M_t} (D \hat{P}_T - K)^+ \right] = \mathbb{E}_t \left[ (F_t + G_t Z_T)^+ \right] \).

- If \( Z_t \) is one-dimensional, option pricing is tractable, as long as the Fourier transform of \( \ln Z_T / Z_t \) is tractable.

- The one-factor \( Z_t \) can be driven by an \( n \)-dimensional cascade stochastic volatility dynamics:
  \[
  \frac{dZ_t}{Z_t} = \sqrt{v^n_t} dW_t, \\
  dv^j_t = \kappa_{vj} (v^{j+1}_t - v^j_t) dt + \sigma_{vj} \sqrt{v^j_t} dW^v_t, \quad j = 1, \ldots, n; \quad v^0 = \theta_v
  \\
  \rho dt = \mathbb{E}[dW_t dW_t^v], \text{ zero correlation among other Brownian pairs.}
  \]

  \( \Rightarrow \) Fourier transform is exponential-affine.

- Alternatively, set \( v_t = v^0 + \tilde{v}_t \), and model the dynamics of \( \tilde{v}_t \) analogous to the dynamics of the bond factors \( x_t \). \( \Rightarrow \) Fourier transform is affine.

- Parsimonious, dimension-invariant specification:
  \[
  \kappa_{vj} = s_v^{n-j} \kappa_v, s_v > 1, \quad \text{power law scaling for frequency distribution}
  \\
  \sigma_{vj} = \sigma_v, \quad \text{iid for shocks of all frequencies}
  \]
Volatility dynamics can be estimated analogous using 70 at-the-money swaption implied volatility series, from a matrix of

- 7 swap maturities: 1, 2, 3, 4, 5, 7, 10 years
- 10 option maturities: 1, 3, 6 months and 1, 2, 3, 4, 5, 7, 10 years

The implied volatility is plugged into the Black formula to generate the invoice price.

PCA analysis on weekly changes in the implied volatilities shows that the first three principal components explain 73.47%, 11.65%, and 3.41% of variation, respectively.

- The loadings of the first principal component are positive on all series $\Rightarrow$ the level factor.
Second and third principal components reflect variations in the slope and curvature along the option maturity dimension. There is little slope/curvature variation along the swap maturity.  

⇒ All interest-rate factors may be driven by the same volatility, but the dynamics of this volatility contain multiple dimensions of variations.
Interest-rate option pricing is easier said than done...

Several issues remain unresolved.

- Accommodate the \( m \) interest-rate factors with a one-dimensional \( Z_t \), or

- Find a positive \((m \times 1)\) vector process \( Z_t \) with tractable density for \( b^\top Z_t \).

- By construction of \( M \), independent volatility risk is not priced...
The linearity-generating dynamics are intriguing.

With such dynamics, one can achieve a true separation of the interest rate term structure and option pricing.

Compared to traditional VAR(1)-type factor specifications, the LG factor dynamics look “unusual.”

- How does the LG dynamics compare with VAR(1) in matching time-series behavior of interest rates?
- Is there any economic support for the LG specification?
  - Alexander David & Pietro Veronesi have endogenized a similar process.

Pricing interest-rate options remain unfinished work ...

Concluding remarks