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Objectives

- How to model financial security returns using *time-changed Lévy processes* with an eye on *data* and *economic reasoning*.

- How to price options based on these models with an eye on *numerical efficiency*.

- How to estimate these models with an eye on *different applications*:
  - market-making,
  - long-term convergence trading,
  - risk-premium taking for systematic risk exposure,
Why time-changed Lévy processes?

Key advantages:

- Generality:
  - Lévy processes can generate almost any return innovation distribution.
  - Applying stochastic time changes randomizes the innovation distribution over time ⇒ stochastic volatility, correlation, skewness, ....

- Explicit economic mapping:
  - Each Lévy component ↔ shocks from one economic source.
  - Time change captures the time-varying intensity of its impact.
  ⇒ makes model design more intuitive, parsimonious, and sensible.

- Tractability: A model is tractable for option pricing if we have
  - tractable characteristic exponent for the Lévy components.
  - tractable Laplace transform for the time change.
  ⇒ Any combinations of the two generate tractable return dynamics.
A Lévy process is a continuous-time process that generates stationary, independent increments ...

Think of return innovation in discrete time: \( R_{t+1} = \mu_t + \sigma_t \varepsilon_{t+1} \).

Lévy processes generate iid return innovation distributions via the Lévy triplet \((\mu, \sigma, \pi(x))\). \((\pi(x) – \text{Lévy density})\).

The Lévy-Khintchine Theorem:

\[
\phi_X(u) \equiv \mathbb{E} \left[ e^{iuX_t} \right] = e^{-t\psi(u)}, \quad u \in \mathcal{D} \subseteq \mathbb{C}
\]

\[
\psi(u) = -iu\mu + \frac{1}{2} u^2 \sigma^2 + \int_{\mathbb{R}_0} \left( 1 - e^{iux} + iux1_{|x|<1} \right) \pi(x)dx,
\]

Innovation distribution \(\leftrightarrow\) characteristic exponent \(\psi(u) \leftrightarrow\) Lévy triplet

- Constraint: \(\int_0^1 x^2 \pi(x)dx < \infty\) (finite quadratic variation).
- Tractable: The integral can be carried out explicitly.
Tractable examples

1. Brownian motion (BSM) \((\mu t + \sigma W_t)\): normal shocks.

2. Compound Poisson jumps (Merton, 76): Large but rare events.

\[
\pi(x) = \lambda \frac{1}{\sqrt{2\pi v_J}} \exp \left( -\frac{(x - \mu_J)^2}{2v_J} \right).
\]

3. Dampened power law (DPL):

\[
\pi(x) = \begin{cases} 
\lambda \exp (-\beta_+ x)x^{-\alpha-1}, & x > 0, \\
\lambda \exp (-\beta_- |x|) |x|^{-\alpha-1}, & x < 0,
\end{cases} \quad \alpha \in [-1, 2), \quad \lambda, \beta \pm > 0
\]

- **Finite activity** when \(\alpha < 0\): \(\int_{\mathbb{R}^0} \pi(x)dx < \infty\). Compound Poisson. Large and rare events.
- **Infinite activity** when \(\alpha \geq 0\): Both small and large jumps.
- **Infinite variation** when \(\alpha \geq 1\): many small jumps,
\[
\int_{\mathbb{R}^0} (|x| \wedge 1) \pi(x)dx = \infty.
\]

\(\alpha \leq 2\) to guarantee finite quadratic variation.

*Market movements of all magnitudes, from small movements to market crashes.*
Analytical characteristic exponents

- **Diffusion**: \( \psi(u) = -iu\mu + \frac{1}{2}u^2\sigma^2 \).

- **Merton's compound Poisson jumps**: \[ \psi(u) = \lambda \left( 1 - e^{iu\mu J - \frac{1}{2}u^2\nu J} \right) . \]

- **Dampened power law**: (for \( \alpha \neq 0, 1 \))
  \[ \psi(u) = -\lambda \Gamma(-\alpha) \left[ (\beta_+ - iu)^\alpha - \beta_+^\alpha + (\beta_- + iu)^\alpha - \beta_-^\alpha \right] - iuC(h) \]
  
  - When \( \alpha \to 2 \), smooth transition to diffusion (quadratic function of \( u \)).
  - When \( \alpha = 0 \) (Variance-gamma by Madan et al):
    \[ \psi(u) = \lambda \ln \left( 1 - iu/\beta_+ \right) \left[ (1 + iu/\beta_-) = \lambda \left( \ln(\beta_+ - iu) - \ln \beta + \ln(\beta_- + iu) - \ln \beta_- \right) . \]
  - When \( \alpha = 1 \) (exponentially dampened Cauchy, Wu 2006):
    \[ \psi(u) = -\lambda \left( (\beta_+ - iu) \ln (\beta_+ - iu) / \beta_+ + \lambda (\beta_- + iu) \ln (\beta_- + iu) / \beta_- \right) - iuC(h). \]
  - \( \beta_\pm = 0 \) (no dampening): \( \alpha \)-stable law
Other Lévy examples

- Other examples:
  - The normal inverse Gaussian (NIG) process of Barndorff-Nielsen (1998)
  - The generalized hyperbolic process (Eberlein, Keller, Prause (1998))
  - The Meixner process (Schoutens (2003))
  - ...

- Bottom line:
  - All tractable in terms of analytical characteristic exponents $\psi(u)$.
  - We can use FFT to generate the density function of the innovation (for model estimation).
  - We can also use FFT to compute option values ...

Question: Do we need Lévy jumps to model financial security returns?

- It is important to look at the data...
Implied volatility smiles & skews on a stock

Moneyness = \ln\left(\frac{K}{F}\right) / \sigma \sqrt{\tau}

Short-term smile

Long-term skew

Maturities: 32 95 186 368 732
Implied volatility skews on a stock index (SPX)

More skews than smiles

Maturities: 32 60 151 242 333 704

Moneyness = \frac{\ln(K/F)}{\sigma \sqrt{\tau}}

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Time-Changed Lévy Processes
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Average implied volatility smiles on currencies

Maturities: 1m (solid), 3m (dashed), 1y (dash-dotted)
(1) The role of jumps at very short maturities

- Implied volatility smiles (skews) ↔ non-normality (asymmetry) for the risk-neutral return distribution (Backus, Foresi, Wu (97)):

\[ IV(d) \approx ATMV \left( 1 + \frac{\text{Skew.}}{6} d + \frac{\text{Kurt.}}{24} d^2 \right), \quad d = \frac{\ln K/F}{\sigma \sqrt{\tau}} \]

- Two mechanisms to generate return non-normality:
  - Use Lévy jumps to generate non-normality for the innovation distribution.
  - Use stochastic volatility to generate non-normality through mixing over multiple periods.

- Over very short maturities (1 period), only jumps contribute to return non-normalities.
The impacts of jumps at very long horizons

- Central limit theorem (CLT): Return distribution converges to normal with aggregation under certain conditions (finite return variance,...) ⇒ As option maturity increases, the smile should flatten.

- Evidence: The skew does not flatten, but steepens!

  - Return variance is infinite. ⇒ CLT does not apply.
  - Down jumps only. ⇒ Option has finite value.

- But CLT seems to hold fine statistically:

![Skewness on S&P 500 Index Return](image1)

![Kurtosis on S&P 500 Index Return](image2)

- Model return innovations under $\mathbb{P}$ by DPL:

\[
\pi(x) = \begin{cases} 
\lambda \exp(-\beta_+ x) x^{-\alpha-1}, & x > 0, \\
\lambda \exp(-\beta_- |x|) |x|^{-\alpha-1}, & x < 0.
\end{cases}
\]

All return moments are finite with $\beta_\pm > 0$. CLT applies.

- Market price of jump risk ($\gamma$):

\[
\left. \frac{dQ}{d\mathbb{P}} \right|_t = \mathcal{E}(-\gamma X)
\]

- The return innovation process remains DPL under $\mathbb{Q}$:

\[
\pi(x) = \begin{cases} 
\lambda \exp(-(\beta_+ + \gamma) x) x^{-\alpha-1}, & x > 0, \\
\lambda \exp(-(\beta_- - \gamma) |x|) |x|^{-\alpha-1}, & x < 0.
\end{cases}
\]

- To break CLT under $\mathbb{Q}$, set $\gamma = \beta_-$ so that $\beta^-_Q = 0$.

- Reconciling $\mathbb{P}$ with $\mathbb{Q}$: Investors pay maximum price on hedging against down jumps.
When a company defaults, its stock value jumps to zero.

This default risk generates a steep skew in long-term stock options.

Evidence: Stock option implied volatility skews are correlated with credit default swap (CDS) spreads written on the same company.

Three Lévy jump components

I. Market risk (FMLS under $\mathcal{Q}$, DPL under $\mathbb{P}$)

II. Idiosyncratic risk (DPL under both $\mathbb{P}$ and $\mathcal{Q}$)

III. Default risk (Compound Poisson jumps).

- Stock options: Information and identification
  - Identify market risk from stock index options.
  - Identify the credit risk component from the CDS market.
  - Identify the idiosyncratic risk from the single-name stock options.

- Currency options:
  - Model currency return as the difference of two log pricing kernels (market risks).
  - Default risk also shows up in FX for low-rating economies.

Economic implications

- In the Black-Scholes world (one-factor diffusion):
  - The market is complete with a bond and a stock.
  - The world is risk free after delta hedging.
  - Utility-free option pricing. Options are redundant.

- In a pure-diffusion world with stochastic volatility:
  - Market is complete with one (or a few) extra option(s).
  - The world is risk free after delta and vega hedging.

- In a world with jumps of random sizes:
  - The market is inherently incomplete (with stocks alone).
  - Need all options (+ model) to complete the market.
  - **Challenges:** Greeks-based dynamic hedging is no longer risk proof.
  - **Opportunities:** Options market is informative/useful:
    - Cross sections \((K, T) \leftrightarrow \mathbb{Q}\) dynamics.
    - Time series \((t) \leftrightarrow \mathbb{P}\) dynamics.
    - The difference \(\mathbb{Q}/\mathbb{P} \leftrightarrow \text{market prices of economic risks.}\)
Beyond Lévy processes

- Lévy processes can generate different iid return innovation distributions.
  - Any distribution you can think of, we can specify a Lévy process, with the increments of the process matching that distribution.

- Yet, return distribution is not iid. It varies over time.
  - That’s why I have shown you only cross-sectional plots ...

- We need to go beyond Lévy processes to capture the time variation in the return distribution (implied volatility surface):
  - Stochastic volatility
  - Stochastic risk reversal (skewness)
  - Predictability of return or volatility.
At-the-money implied volatilities at fixed time-to-maturities from 1 month to 5 years.
Three-month delta-neutral straddle implied volatility.
Stochastic skewness on stock indexes

Implied volatility spread between 80% and 120% strikes at fixed time-to-maturities from 1 month to 5 years.
Three-month 10-delta risk reversal (blue lines) and butterfly spread (red lines).
Randomize the time

- Review the Lévy-Khintchine Theorem:

\[
\phi(u) \equiv \mathbb{E}[e^{iuX_t}] = e^{-t\psi(u)}, \\
\psi(u) = -iu\mu + \frac{1}{2}u^2\sigma^2 + \lambda \int_{\mathbb{R}_0} (1 - e^{iu} + iux1_{|x|<1}) \tilde{\pi}(x) dx,
\]

- The drift \( \mu \), the diffusion variance \( \sigma^2 \), and the mean arrival rate \( \lambda \) are all proportional to time \( t \).

- We can directly specify \((\mu_t, \sigma^2_t, \lambda_t)\) as following stochastic processes.

- Or we can randomize time \( t \rightarrow T_t \) for the same result.

- We define \( T_t \equiv \int_0^t \nu_s^- ds \) as the *stochastic time change*, with \( \nu_t \) being the *instantaneous activity rate*.

- Depending on the Lévy specification, the activity rate has the same meaning (up to a scale) as a randomized version of the *instantaneous drift*, *instantaneous variance*, or *instantaneous arrival rate*.
In 1949, Bochner introduced the notion of time change to stochastic processes. In 1973, Clark suggested that time-changed diffusions could be used to accurately describe financial time series.

Ane & Geman (2000) show supporting evidence: Define returns over fixed number of trades, not over fixed calendar time intervals.

Two types of clocks can be used to model business time:

1. Clocks based on increasing jump processes have staircase like paths.
2. Continuous clocks ($T_t \equiv \int_0^t \nu_s^- \, ds$) as we have just defined.

The first type of clock can transform a diffusion into a jump process — All Lévy processes considered earlier can be generated as changing the clock of a diffusion with an increasing jump process (subordinator).

The second type of business clock can be used to describe stochastic volatility (and higher moments).

Monroe (1978): All semimartingales can be generated by applying stochastic time changes (of both types) on Brownian motions.
Economic interpretations

- Treat \( t \) as the calendar time, and \( T_t \equiv \int_0^t \nu_s \, ds \) as the **business time**.
  - Business activity accumulates with calendar time, but the speed varies, depending on the business activity.
  - Business activity tends to intensify before earnings announcements, FOMC meeting days...
  - In this sense, \( \nu_t \) captures the intensity of business activity at time \( t \).
  - This interpretation has inspired many microstructure works...

- **Economics shocks (impulse) and financial market responses:**
  - Think of each Lévy process (component) as capturing one source of economic shock.
  - The stochastic time change on each Lévy component captures the random intensity of the impact of the economic shock on the financial security.

\[
\text{Return} \sim \sum_{i=1}^{K} X_{T_t}^i \sim \sum_{i=1}^{K} (\text{Economic shock})^i_{\text{Stochastic impact}}.
\]
Example: Return on a stock

- Model the return on a stock to reflect shocks from two sources:
  - **Credit risk**: In case of corporate default, the stock price falls to zero. Model the impact as a Poisson Lévy jump process with log return jumps to negative infinity upon jump arrival.
  - **Market risk**: Daily market movements (small or large). Model the impact as a diffusion or infinite-activity (infinite variation) Lévy jump process or both.

- Apply separate time changes to the two Lévy components to capture (1) the intensity variation of corporate default, (2) the market risk (volatility) variation.

- **Key**: *Each component has a specific economic purpose.*
Example: A CAPM model:

\[
\ln \frac{S_t^j}{S_0^j} = (r - q) t + \left( \beta^j X_{T_t}^m - \varphi_{X}^m(\beta^j) T_t^m \right) + \left( X_{T_t}^j - \varphi_{X}(1) T_t^j \right). 
\]

- Estimate \( \beta \) and market prices of return and volatility risk using index and single name options.
- Cross-sectional analysis of the estimates.

An international CAPM:

Example: Return on an exchange rate

- Exchange rate reflects the interaction between two economic forces.

- Use two Lévy processes to model the two economic forces separately.

- Consider a negatively skewed distribution (downside jumps) from each economic source (crash-o-phobia from both sides). Use the difference to model the currency return between the two economies.

- Apply separate time changes to the two Lévy processes to capture the strength variation (tug war) between the two economic forces.

  - Stochastic time changes on the two negatively skewed Lévy processes generate both stochastic volatility and stochastic skew.

- Key: Each component has its specific economic purpose.

Example: Exchange rates and pricing kernels

- Exchange rate reflects the interaction between two economic forces.
- The economic meaning becomes clearer if we model the pricing kernel of each economy.
  - Let $m_{0,t}^{US}$ and $m_{0,t}^{JP}$ denote the pricing kernels of the US and Japan. Then the dollar price of yen $S_t$ is given by
    \[
    \ln \frac{S_t}{S_0} = \ln m_{0,t}^{JP} - \ln m_{0,t}^{US}.
    \]
  - If we model the negative of the logarithm of each pricing kernel ($-\ln m_{0,t}^j$) as a time-changed Levy process, $X_{T_t}^j$ ($j = US, JP$) with negative skewness. Then, $\ln \frac{S_t}{S_0} = \ln m_{0,t}^{JP} - \ln m_{0,t}^{US} = X_{T_t}^{US} - X_{T_t}^{JP}$
    - Think of $X$ as consumption growth shocks
    - Think of $T_t$ as time-varying risk premium.

- Consistent and simultaneous modeling of all currency pairs.

To compute the time-0 price of a European option price with expiry at $t$, we first compute the Fourier transform of the log return $s_t \equiv \ln S_t / S_0$.

The generalized Fourier transform of a time-changed Lévy process:

$$
\phi_Y(u) \equiv \mathbb{E}^Q [e^{iuX_{T_t}}] = \mathbb{E}^M [e^{-\psi_x(u)T_t}], \quad u \in \mathcal{D} \subset \mathbb{C},
$$

where the new measure $\mathbb{M}$ is defined by the exponential martingale:

$$
\frac{d\mathbb{M}}{d\mathbb{Q}} \bigg|_t = \exp (iuX_{T_t} + T_t \psi_x(u)).
$$

Without time-change, $e^{iuX_t + t\psi_x(u)}$ is an exponential martingale by Lévy-Khintchine Theorem.

A continuous time change does not change the martingality.

$\mathbb{M}$ is complex valued (no longer a probability measure).

Tractability of the transform $\phi(u)$ depends on the tractability of

- The characteristic exponent of the Lévy process $\psi_x(u)$.
- The Laplace transform of $T_t$ under $\mathbb{M}$.

$(X, T_t)$ can be chosen separately as building blocks to capture the two dimensions: Moneyness & term structure.
The Laplace transform of the stochastic time $\mathcal{T}_t$

- We have solved the characteristic exponent of the Lévy process (by the Lévy-Khintchine Theorem).
- Compare the Laplace transform of the stochastic time,

$$\mathcal{L}_{\mathcal{T}}(\psi) \equiv \mathbb{E} \left[ e^{-\psi \mathcal{T}_t} \right] = \mathbb{E} \left[ e^{-\psi \int_0^t \nu_s ds} \right]$$  \hspace{1cm} (1)

- to the pricing equation for zero-coupon bonds:

$$B(0, t) \equiv \mathbb{E}^Q \left[ e^{-\int_0^t r_s ds} \right]$$  \hspace{1cm} (2)

- The two pricing equations look analogous
  - Both $\nu_t$ and $r_t$ need to be positive.
  - If we set $r_t = \psi \nu_t$, $\mathcal{L}_{\mathcal{T}}(\psi)$ is essentially the bond price.

The analogy allows us to borrow the vast bond pricing literature:
- **Affine class**: Zero-coupon bond prices are exponential affine in the state variable.
- **Quadratic**: Zero-coupon bond prices are exponential quadratic in the state variable.
- ...
With the Fourier transform of the log return \((\phi(u))\), we can compute vanilla option values via Fourier inversion.

Take a European call option as an example.

Perform the following rescaling and change of variables:

\[
c(k) = e^{rt} c(K, t)/F_0 = \mathbb{E}_0^Q \left[ (e^{s_t} - e^k) 1_{s_t \geq k} \right],
\]

with \(s_t = \ln F_t/F_0\) and \(k = \ln K/F_0\).

- \(c(k)\): the option forward price in percentage of the underlying forward as a function of moneyness defined as the log strike over forward, \(k\) (at a fixed time to maturity).

Derive the Fourier transform of the scaled option value \(c(k) (\chi_c(u))\) in terms of the Fourier transform \((\phi_s(u))\) of the log return \(s_t = \ln F_t/F_0\).

Perform numerical Fourier inversion to obtain option value.

There are two ways of doing this.
I. The CDF analog

- Treat \( c(k) = \mathbb{E}_0^Q \left[ (e^{s_t} - e^k) \mathbf{1}_{s_t \geq k} \right] = \int_{-\infty}^{\infty} (e^{s_t} - e^k) \mathbf{1}_{s_t \geq x} dF(s) \) as a CDF.

- The option transform:
  \[
  \chi_c^l(u) \equiv \int_{-\infty}^{\infty} e^{iku} dc(k) = -\frac{\phi_s(u - i)}{iu + 1}, \quad u \in \mathbb{R}.
  \]

- The inversion formula is analogous to the inversion of a CDF:
  \[
  c(x) = \frac{1}{2} + \frac{1}{2\pi} \int_{0}^{\infty} \frac{e^{iux} \chi_c^l(-u) - e^{-iux} \chi_c^l(u)}{iu} du.
  \]

- Use quadrature methods for the numerical integration.
  - It can work well if done right.

**References:** Duffie, Pan, Singleton, 2000, Transform Analysis and Asset Pricing for Affine Jump Diffusions, Econometrica, 68(6), 1343–1376.

II. The PDF analog

- Treat $c(k)$ analogous to a PDF. (Carr and Madan (1999), Carr & Wu (2004), ...)
  - The option transform:
    \[
    \chi_{II}^{c}(z) \equiv \int_{-\infty}^{\infty} e^{izk} c(k) dk = \frac{\phi_s (z - i)}{(iz)(iz + 1)}
    \]
    with $z = u - i\alpha$, $\alpha \in \mathcal{D} \subseteq \mathbb{R}^+$ for the transform to be well defined.
    - The range of $\alpha$ depends on payoff structure and model.
    - The exact choice of $\alpha$ is a numerical issue (asking for more research...)

- The inversion is analogous to that for a PDF:
  \[
  c(k) = \frac{1}{2} \int_{-i\alpha - \infty}^{-i\alpha + \infty} e^{-izk} \chi_{II}^{c}(z) dz = \frac{e^{-\alpha k}}{\pi} \int_{0}^{\infty} e^{-iu k} \chi_{II}^{c}(u - i\alpha) du.
  \]
  - The numerical integration can be cast into an FFT to improve the computational speed.
  - Use fractional FFT to separate the choice of strike grids from the integration grids (Chourdakis (2005)).
    - Room for improvement...
Estimating statistical dynamics

- For Lévy processes **without** time change, maximum likelihood estimation: 
  - Given initial parameters guess, derive the return characteristic function.
  - Apply FFT to generate the probability density at a fine grid of possible return realizations.
  - Interpolate to obtain the density at the observed return values.
  - Numerically maximize the aggregate log likelihood.

- For time-changed Lévy processes with **observable** activity rates, it is still straightforward to apply MLE.

- For time-changed Lévy processes with **hidden** activity rates, some filtering technique is needed to infer the hidden states from the observable.
  - Maximum likelihood with partial filtering: Alireza Javaheri
  - MCMC Bayesian estimation: Eraker, Johannes, Polson (2003, JF), Li, Wells, Yu, (RFS)

- Use more data (and transformation) to turn hidden states into observable quantities. Wu (2007), Aït-Sahalia and Robert Kimmel (2007), Bondarenko (2007)…
Estimating risk-neutral dynamics

- **Daily fitting**: (Bakshi, Cao, Chen (1997, JF), Carr and Wu (2003, JF))
  - Nonlinear weighted least square to fit models to option prices.
  - Parameters and state variables (activity rates) are treated as the same.
  - *What to hedge*: state variables or parameters or both.
  - Can experience identification issues for sophisticated models.
  - Better applied to Lévy processes without time change.

- **Dynamically consistent estimation**:
  - Parameters are fixed, only activity rates are allowed to vary over time.
  - Numerically more challenging.
  - Better applied to more sophisticated models that perform well over different market conditions.
Static v. dynamic consistency

- **Static cross-sectional consistency**: Option values across different strikes/maturities are generated from the same model (same parameters) at a point in time.

- **Dynamic consistency**: Option values over time are also generated from the same no-arbitrage model (same parameters).

Different needs for different market participants:

- **Market makers**:
  - Achieving static consistency is sufficient.
  - Matching market prices is important to provide two-sided quotes.

- **Long-term convergence traders**:
  - Dynamic consistency is important.
  - A good model should generate large (you wish) but highly convergent pricing errors, and provide robust hedging ratios.

A well-designed model (with several time-changed Lévy components) can achieve both dynamic consistency and good performance.
Dynamically consistent estimation

- Nested nonlinear least square (Huang & Wu (2004), Bates (2000)):
  Often has convergence issues.

- Cast the model into state-space form and use MLE (Carr & Wu (2007a,b), Bakshi, Carr, Wu (2008), Mo & Wu (2007), Heidari & Wu (2008), Leippold & Wu (2007),...)
  - Define state propagation equation based on the \( \mathbb{P} \)-dynamics of the activity rates. (Need to specify market price on activity rates).
  - Define the measurement equation based on option prices (out-of-money values, weighted by vega,...)
  - Use an extended version of Kalman filter (EKF, UKF, PKF) to predict/filter the distribution of the states and measurements.
  - Define the likelihood function based on forecasting errors on the measurement equations.
  - Estimate model parameters by maximizing the likelihood.
Joint estimation of $P$ and $Q$ dynamics

Important in learning investors’ risk-taking behaviors.

- Pan (2002): GMM.
- Eraker (2004): Bayesian with MCMC. Choose 2-3 options per day!
- Bakshi & Wu (2005), “Investor Irrationality and the Nasdaq Bubble”

MLE with filtering

- Cast activity rate $P$-dynamics into state equation, cast option prices into measurement equation.
- Use UKF to filter out the mean and covariance of the states and measurement.
- Construct the likelihood function of options based on forecasting errors (from UKF) on the measurement equations.
- Given the filtered activity rates, construct the conditional likelihood on the Nasdaq-100 index returns by FFT inversion of the conditional characteristic function.
- The joint log likelihood equals the sum of the log likelihood of option pricing errors and the conditional log likelihood of index returns.
Concluding remarks

- Modeling security returns with time-changed Lévy processes enjoys three key virtues: (1) Generality; (2) explicit economic mapping; (3) tractability.

- The framework provides a nice starting point for generating security return dynamics that are parsimonious, tractable, economically sensible, and statistically performing well.

- Going beyond time changed Lévy processes:
  - More economics: Link the dynamics to firm characteristics, capital structure decisions.
  - More market microstructure: How to accommodate inventory, order flow into the dynamics.