Lévy Processes and Option Pricing

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“Incorporating jumps using jump-diffusions and Lévy processes?”

- A Lévy process is a continuous-time process that generates stationary, independent increments ...

- Think of return innovations \( (\varepsilon) \) in discrete time:
  \[ R_{t+1} = \mu t + \sigma_t \varepsilon_{t+1}. \]
  - Normal return innovation — diffusion
  - Non-normal return innovation — jumps

- Classic Lévy specifications in finance:
  - Brownian motion (Black-Scholes)
  - Compound Poisson process with normal jump size (Merton)

⇒ The return innovation distribution is either normal or mixture of normals.
Lévy processes greatly expand our continuous-time choices of iid return innovation distributions via the Lévy triplet \((\mu, \sigma, \pi(x))\). (\pi(x)–Lévy density).

The Lévy-Khintchine Theorem:

\[
\phi_{X_t}(u) \equiv \mathbb{E}[e^{iuX_t}] = e^{-t\psi(u)},
\]
\[
\psi(u) = -iu\mu + \frac{1}{2}u^2\sigma^2 + \int_{\mathbb{R}_0} (1 - e^{iux} + iux1_{|x|<1}) \pi(x)dx,
\]

Innovation distribution
\leftrightarrow \text{characteristic exponent } \psi(u)
\leftrightarrow \text{Lévy triplet } (\mu, \sigma, \pi(x))

- Constraint: \(\int_0^1 x^2\pi(x)dx < \infty\).
- “Tractable:” if the integral can be carried out explicitly.
Tractable examples

- Brownian motion \((\mu t + \sigma W_t)\): normal shocks.
- Compound Poisson jumps: Large but rare events.

\[
\pi(x) = \lambda \frac{1}{\sqrt{2\pi v_J}} \exp\left(-\frac{(x - \mu_J)^2}{2v_J}\right).
\]

- Dampened power law (DPL):

\[
\pi(x) = \begin{cases} 
  \lambda \exp(-\beta_+x) x^{-\alpha-1}, & x > 0, \\
  \lambda \exp(-\beta_-|x|) |x|^{-\alpha-1}, & x < 0, 
\end{cases}
\]

\[
\lambda, \beta_\pm > 0, \quad \alpha \in [-1, 2)
\]

- Finite activity when \(\alpha < 0\): \(\int_{\mathbb{R}^0} \pi(x) \, dx < \infty\). Compound Poisson. Large and rare events.
- Infinite activity when \(\alpha \geq 0\): Both small and large jumps. Jump frequency increases with declining jump size, and approaches infinity as \(x \to 0\).
- Infinite variation when \(\alpha \geq 1\): many small jumps.

*Market movements of all magnitudes, from small movements to market crashes.*
Analytical characteristic exponents

- **Diffusion**: $\psi(u) = -iu\mu + \frac{1}{2}u^2\sigma^2$.

- **Merton's compound Poisson jumps**: $\psi(u) = \lambda \left( 1 - e^{iu\mu J - \frac{1}{2}u^2v_J} \right)$.

- **Dampened power law**: (for $\alpha \neq 0, 1$)
  
  $\psi(u) = -\lambda \Gamma(-\alpha) \left[ (\beta_+ - iu)^\alpha - \beta^\alpha_+ + (\beta_- + iu)^\alpha - \beta^\alpha_- \right] - iuC(h)$

  - When $\alpha \to 2$, smooth transition to diffusion (quadratic function of $u$).
Other Lévy examples

- Other examples:
  - The normal inverse Gaussian (NIG) process of Barndorff-Nielsen (1998)
  - The generalized hyperbolic process (Eberlein, Keller, Prause (1998))
  - The Meixner process (Schoutens (2003))
  - ...

- Bottom line:
  - All tractable in terms of analytical characteristic exponents $\psi(u)$.
  - We can use FFT to generate the density function of the innovation (for model estimation).
  - We can also use FFT to compute option values.
Run Brownian motions on a business clock

- Clark (1973): If one runs a Brownian motion on a business clock, the resulting process matches financial time series better.

- The possibility that business clock may not move while calendar time marches forward is important ...
  - If the clock is a standard Poisson process
    - The resulting process is a compound Poisson process with normal jump sizes.
  - If the clock is a compound Poisson process with exponentially distributed jump size
    - DPL with $\alpha = -1$
  - If the clock is a gamma process
    - DPL with $\alpha = 0$.
  - If the clock is continuous
    - a continuous process.
General evidence on Lévy return innovations

- **Credit risk:** (compound) Poisson process
  - The whole intensity-based credit modeling literature...

- **Market risk:** Infinite-activity jumps
  - Evidence from stock returns (CGMY (2002)): The $\alpha$ estimates for DPL on most stock return series are greater than zero.
  - Evidence from options: Models with infinite-activity return innovations price equity index options better (Carr and Wu (2003), Huang and Wu (2004))
Implied volatility smiles & skews on a stock

- Short-term smile
- Long-term skew

Maturities: 32 95 186 368 732

Moneyness = \frac{\ln(K/F)}{\sigma \sqrt{\tau}}
Implied volatility skews on a stock index (SPX)

More skews than smiles

Maturities: 32  60  151  242  333  704

Moneyness = \frac{\ln(K/F)}{\sigma \sqrt{\tau}}
Average implied volatility smiles on currencies

Maturities: 1m (solid), 3m (dashed), 1y (dash-dotted)
(I) The role of jumps at very short maturities

- Implied volatility smiles (skews) ↔ non-normality (asymmetry) for the risk-neutral return distribution.

\[ IV(d) \approx ATMV \left( 1 + \frac{\text{Skew.}}{6} d + \frac{\text{Kurt.}}{24} d^2 \right), \quad d = \frac{\ln K/F}{\sigma \sqrt{\tau}} \]

- Two mechanisms to generate return non-normality:
  - Use Lévy jumps to generate non-normality for the innovation distribution.
  - Use stochastic volatility to generate non-normality through mixing over multiple periods.

- Over very short maturities (1 period), *only jumps contribute to return non-normalities.*
Time decay of short-term OTM options

- As option maturity ↓ zero, OTM option value ↓ zero.
- The speed of decay is exponential $\mathcal{O}(e^{-c/T})$ under pure diffusion, but linear $\mathcal{O}(T)$ in the presence of jumps.
- Term decay plot: $\ln(OTM/T) \sim \ln(T)$ at fixed $K$:

In the presence of jumps, the Black-Scholes implied volatility for OTM options $IV(\tau,K)$ explodes as $\tau \downarrow 0$. 
(II) The impacts of jumps at very long horizons

- Central limit theorem (CLT): As option maturity increases, the smile should flatten.
- Evidence: The skew does not flatten, but steepens!
- FMLS: Maximum negatively skewed $\alpha$-stable process.
  - Return variance is infinite. Hence, CLT does not apply.
  - Down jumps only. $\Rightarrow$ Option has finite value.
- But CLT seems to hold fine statistically:

![Skewness on S&P 500 Index Return](image1)

![Kurtosis on S&P 500 Index Return](image2)
Reconcile $\mathbb{P}$ with $\mathbb{Q}$ via DPL jumps

- Model return innovations under $\mathbb{P}$ by DPL:

$$\pi(x) = \begin{cases} 
\lambda \exp(-\beta_+ x)x^{-\alpha-1}, & x > 0, \\
\lambda \exp(-\beta_- |x|)|x|^{-\alpha-1}, & x < 0.
\end{cases}$$

All return moments are finite with $\beta_+ > 0$. \textit{CLT applies.}

- Market prices of jump risks ($\gamma$)

$$\frac{d\mathbb{Q}}{d\mathbb{P}}|_t = \exp(-\gamma X + \text{convexity adjustment})$$

- The return innovation process remains DPL under $\mathbb{Q}$:

$$\pi(x) = \begin{cases} 
\lambda \exp(- (\beta_+ + \gamma) x)x^{-\alpha-1}, & x > 0, \\
\lambda \exp(- (\beta_- - \gamma) |x|)|x|^{-\alpha-1}, & x < 0.
\end{cases}$$

- To break CLT under $\mathbb{Q}$, set $\gamma = \beta_-$ so that $\beta_-^\mathbb{Q} = 0$.

- Reconciling $\mathbb{P}$ with $\mathbb{Q}$: \textit{Investors charge maximum allowed market price on down jumps.}
(III) Default risk & long-term implied vol skew

- When a company defaults, its stock value jumps to zero.
- It generates a steep skew in long-term stock options.
- Evidence: Stock option implied volatility skews are correlated with credit default swap (CDS) spreads written on the same company.
Three Lévy jump components

I. Market risk (FMLS under $\mathbb{Q}$, DPL under $\mathbb{P}$)

II. Idiosyncratic risk (DPL under both $\mathbb{P}$ and $\mathbb{Q}$)

III. Default risk (Compound Poisson jumps).

Remarks:
- Identify market risk from SPX or QQQQ options.
- Identify the credit risk component from the credit default swap (CDS) market.

Currency options:
- Model currency returns as the difference of two log pricing kernels (market risks).
- Default risk also shows up in FX for low-rating economies.
Beyond Lévy processes

- Lévy processes can be used to generate different iid return innovation distributions.

- Yet, return distribution is not iid. It varies stochastically over time.

- We need to go beyond Lévy processes to capture the stochastic nature of the return distribution.

- Applying separate time changes to different Lévy components generates
  - separate stochastic responses to each economic shock.
  - stochastic volatility, skewness, ...
At-the-money implied volatilities at fixed time-to-maturities from 1 month to 5 years.
Stochastic volatility on currencies

Three-month delta-neutral straddle implied volatility.
Implied volatility spread between 80% and 120% strikes at fixed time-to-maturities from 1 month to 5 years.
Stochastic skewness on currencies

Three-month 10-delta risk reversal (blue lines) and butterfly spread (red lines).
Economic implications of using jumps

- In the Black-Scholes world (one-factor diffusion):
  - The market is complete with a bond and a stock.
  - The world is risk free after delta hedging.
  - Utility-free option pricing. Options are redundant.

- In a pure-diffusion world with stochastic volatility:
  - Market is complete with one (or a few) extra option(s).
  - The world is risk free after delta and vega hedging.

- In a world with jumps of random sizes:
  - The market is inherently incomplete (with stocks alone).
  - Need all options (+ model) to complete the market.
  - Derman: “Beware of economists with Greek symbols!”
  - Options market is informative/useful:
    - Cross sections \((K, T) \leftrightarrow \mathbb{Q}\) dynamics.
    - Time series \((t) \leftrightarrow \mathbb{P}\) dynamics.
    - The difference \(\mathbb{Q}/\mathbb{P}\) \leftrightarrow market prices of economic risks.
Different types of jumps affect option pricing at both short and long maturities.

- Implied volatility smiles at very short maturities can only be accommodated by a jump component.
- Implied volatility skews at very long maturities ask for a jump process that generates infinite variance.
- Credit risk exposure may also help explain the long-term skew on single name stock options.

The choice of jump types depends on the events:

- Infinite-activity jumps ⇔ frequent market order arrival.
- Finite-activity Poisson jumps ⇔ rare events (credit).

The presence of jumps of random sizes creates value for the options markets ...