Modeling Financial Security Returns Using Lévy Processes

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Overview

Key advantages of modeling security returns with time-changed Lévy processes:

- **Generality:**
  - Lévy processes can generate pretty much any return innovation distribution.
  - Applying stochastic time changes on Lévy processes randomizes the return innovation distribution over time $\Rightarrow$ stochastic volatility, skewness, ....

- **Explicit economic mapping** by modeling returns with several time-changed Lévy components (versus models with hidden state vectors):
  - Each Lévy component captures shocks from one economic source.
  - Time change captures the time-varying intensity of its impact.
  $\Rightarrow$ makes model design more intuitive, parsimonious, and sensible.

- **Tractability:** A model is tractable for option pricing if we have under the risk-neutral measure $\mathbb{Q}$:
  - tractable characteristic exponent for the Lévy components.
  - tractable Laplace transform for the activity rates underlying the time change.
  $\Rightarrow$ any combinations of the two generate tractable return dynamics.
I. Modeling return innovation using Lévy processes
Lévy processes and innovation distributions

- In discrete time, we can assume an arbitrary distribution for the return innovation:

\[ R_{t+1} = \mu_t + \sigma_t \varepsilon_{t+1}. \]

- In continuous time, until recently, we had modeled return innovation using:
  - either a Brownian motion (Black-Scholes)
  - or a compound Poisson process with normal jump size (Merton).
  \[ \Rightarrow \] The return innovation distribution is either normal or mixture of normals.

- Lévy processes greatly expand our continuous-time choice of iid return innovation distributions via the Lévy triplet \((\mu, \sigma, \pi(x))\). \((\pi(x)\)–Lévy density).

- The Lévy-Khintchine Theorem:

\[
\phi_{X_t}(u) \equiv \mathbb{E}[e^{iuX_t}] = e^{-t\psi(u)},
\]

\[
\psi(u) = -iu\mu + \frac{1}{2}u^2\sigma^2 + \int_{\mathbb{R}} (1 - e^{iux} + iux1_{|x|<1}) \pi(x)dx,
\]

distribution \(\leftrightarrow\) characteristic exponent \(\psi(u) \leftrightarrow\) Lévy triplet \((\mu, \sigma, \pi(x))\)

- Constraint: \(\int_0^1 x^2 \pi(x)dx < \infty\) (finite quadratic variation).

- The model is “tractable” if the integral in \(\psi(u)\) can be carried out explicitly.
Tractable examples

- Brownian motion \((\mu t + \sigma W_t)\): *Continuous movements; normal shocks.*

- Merton’s compound Poisson jumps: *Large but rare events, corporate default.*

\[
\pi(x) = \lambda \frac{1}{\sqrt{2\pi v_J}} \exp\left( -\frac{(x - \mu_J)^2}{2v_J} \right). 
\] (1)

- Dampened power law (CGMY, Wu):

\[
\pi(x) = \begin{cases} 
\lambda \beta_+ \exp(-\beta_+ x) x^{-\alpha-1}, & x > 0, \\
\lambda \beta_- \exp(-\beta_- |x|) |x|^{-\alpha-1}, & x < 0,
\end{cases} 
\]

\[
\lambda, \beta_+, \beta_- > 0, \alpha \in [-1, 2). 
\] (2)

- *Finite activity* when \(\alpha < 0\): \(\int_{\mathbb{R}_0^+} \pi(x) dx < \infty\). Large but rare events.

- *Infinite activity* when \(\alpha \geq 0\): Both small and large jumps. Jump frequency increase with declining jump size, and approaches infinity as \(x \to 0\).

- *Infinite variation* when \(\alpha \geq 1\): many small jumps.

*Market movements of all magnitudes, from small movements to market crashes.*
Analytical characteristic exponents

- **Diffusion:** \( \psi(u) = -iu\mu + \frac{1}{2}u^2\sigma^2. \)

- **Merton’s compound Poisson jumps:**
  \[
  \psi(u) = \lambda \left( 1 - e^{iu\mu J - \frac{1}{2}u^2\nu J} \right).
  \]

- **Dampened power law:**
  \[
  \psi(u) = -\Gamma(-\alpha)\lambda \left[ (\beta_+ - iu)^\alpha - \beta_+^\alpha + (\beta_- + iu)^\alpha - \beta_-^\alpha \right] - iuC(h)
  \]  
  for \( \alpha \neq 0, \alpha \neq 1. \)

  - When \( \alpha \to 2, \) smooth transition to diffusion (quadratic function of \( u \)).
  - When \( \alpha = 0 \) (Variance-gamma by Madan et al):
    \[
    \psi(u) = \lambda \ln \left( 1 - iu/\beta_+ \right) \left( 1 + iu/\beta_- \right) = \lambda \left( \ln(\beta_+ - iu) - \ln \beta + \ln(\beta_- + iu) - \ln \beta \right)
    \]
  - When \( \alpha = 1 \) (exponentially dampened Cauchy, Wu 2006):
    \[
    \psi(u) = -\lambda \left( (\beta_+ - iu) \ln (\beta_+ - iu) / \beta_+ + \lambda (\beta_- + iu) \ln (\beta_- + iu) / \beta_- \right) - iuC(h).
    \]
Other Lévy examples

- The normal inverse Gaussian (NIG) process of Barndorff-Nielsen (1998)
- The generalized hyperbolic process (Eberlein, Keller, Prause (1998))
- The Meixner process (Schoutens (2003))

- All tractable in terms of the characteristic exponents $\psi(u)$.
- We can use FFT to generate the density function of the innovation (for model estimation).
Evidence on Lévy return innovations

- **Credit risk:** *(compound) Poisson process*
  - The whole intensity-based credit modeling literature...
  - Carr and Wu: The impact of corporate default on stock price: Poisson arrival with jump to zero.
  - Carr and Wu: The impact of sovereign default on currency price: Poisson arrival with random downside jump size distribution.

- **Market risk:** *Infinite-activity jumps*
  - Evidence from stock returns (CGMY (2002)): The $\alpha$ estimates for DPL on most stock return series are greater than zero.
  - Evidence from options: Models with infinite-activity return innovations price equity index options better (Carr and Wu (2003), Huang and Wu (2004))
  - Li and Yu (2005): Infinite-activity jumps cannot be approximated by finite-activity jumps.

- The role of diffusion (in the presence of infinite-variation jumps)
  - Not big, difficult to identify (CGMY (2002), Carr and Wu (2003a,b)).
  - Generate correlations with diffusive activity rates (Huang and Wu (2004)).
  - The diffusion component ($\sigma^2$) is identifiable even in presence of infinite-variation jumps (Aït-Sahalia (2004), Aït-Sahalia&Jacod 2005).
II. Capturing stochastic volatility via time changes
Modeling stochastic volatility

- Discrete-time analog again: $R_{t+1} = \mu_t + \sigma_t \varepsilon_{t+1}$
  - $\varepsilon_{t+1}$ is an iid return innovation, with an arbitrary distribution assumption $\leftrightarrow$ Lévy process.
  - $\sigma_t$ is the conditional volatility, $\mu_t$ is the conditional mean return, both of which can be time-varying, stochastic...

- In continuous time, how do we model stochastic mean/volatility tractably?
  - If the return innovation is modeled by a Brownian motion, we can let the instantaneous variance to be stochastic and tractable, not volatility (Heston(1993), Bates (1996)).
  - If the return innovation is modeled by a compound Poisson process, we can let the Poisson arrival rate to be stochastic, not the mean jump size, jump distribution variance (Bates(2000), Pan(2002)).
  - If the return innovation is modeled by a general Lévy process, it is tractable to randomize the time, or something proportional to time.

*Variance of a Brownian motion, intensity of a Poisson process are both proportional to time.*
Randomize the time

- Review the Lévy-Khintchine Theorem:

\[
\phi(u) \equiv \mathbb{E} \left[ e^{iuX_t} \right] = e^{-t\psi(u)},
\]

\[
\psi(u) = -iu\mu + \frac{1}{2}u^2\sigma^2 \text{ for diffusion with drift } \mu \text{ and variance } \sigma^2,
\]

\[
\psi(u) = \lambda \left( 1 - e^{iu\mu J - \frac{1}{2}u^2v_J} \right) \text{ for Merton's compound Poisson jump.}
\]

- The drift \( \mu \), the diffusion variance \( \sigma^2 \), and the Poisson arrival rate \( \lambda \) are all proportional to time \( t \).

- We may as well randomize time \( t \to T_t \) instead of \( (\mu, \sigma^2, \lambda) \), for the same result.

- We define \( T_t \equiv \int_0^t \nu_s \, ds \) as the (stochastic) time change, with \( \nu_t \) being the instantaneous activity rate.

  - Depending on the Lévy specification, it has the same meaning (up to a scale) as a randomized version of the instantaneous drift, instantaneous variance, or instantaneous arrival rate.
Applying separate time changes

... to different Lévy components

- Consider a Lévy process $X_t \sim (\mu, \sigma^2, \lambda p(x))$.
  - If we apply random time change to $X_t \rightarrow X_{T_t}$ with $T_t = \int_0^t v_s ds$, it is equivalent to assuming that $(\mu_t, \sigma^2_t, \lambda_t)$ are all time varying, but they are all proportional to one common source of variation $v_t$.
  - Suppose we want $(\mu_t, \sigma^2_t, \lambda_t)$ to vary separately, then we need to apply separate time changes to the three Lévy components.
    - Decompose $X_t$ into three Lévy processes: $X^1_t \sim (\mu, 0, 0)$, $X^2_t \sim (0, \sigma^2, 0)$, and $X^3_t \sim (0, 0, \lambda p(x))$, and then apply separate time changes to the three Lévy processes.

- In practice, *we can use one Lévy process to model one source of economic shock, and use separate time changes on different Lévy processes to capture the intensity variation of different economic shocks.*

$$\text{Return} \sim \sum_{i=1}^{K} X^i_{T_t}.$$
Example: Return on a stock

- Model the return on a stock as reflecting shocks from **two sources**:
  - **Credit risk**: In case of corporate default, the stock price falls to zero. Model the impact as a Poisson Lévy jump process with log return jumps to negative infinity upon jump arrival.
  - **Market risk**: Daily market movements (small or large). Model the impact as a diffusion or infinite-activity (infinite variation) Lévy jump process or both.

- Apply separate time changes to the two Lévy components to capture (1) the intensity variation of corporate default, (2) the market risk (volatility) variation.

- **Key**: *Each component has a specific economic purpose.*

Example: Return on an exchange rate

- Exchange rate reflects the interaction between two economic forces.

- Use two Lévy processes to model the two economic forces separately.

- Consider a negatively skewed distribution (downside jumps) from each economic source (crash-o-phobia from both sides). Use the difference to model the currency return between the two economies.

- Apply separate time changes to the two Lévy processes to capture the strength variation of the two economic forces.
  - Stochastic time changes on the two negatively skewed Lévy processes generate both stochastic volatility and stochastic skew.

- Key: Each component has its specific economic purpose.

Exchange rates and pricing kernels

- Exchange rate reflects the interaction between two economic forces.
- The economic meaning becomes clearer if we model the pricing kernel of each economy.
  - Let $m_{0,t}^{US}$ and $m_{0,t}^{JP}$ denote the pricing kernels of the US and Japan. Then the dollar price of yen $S_t$ is given by
    \[ \ln S_t / S_0 = \ln m_{0,t}^{JP} - \ln m_{0,t}^{US}. \]
  - If we model the negative of the logarithm of each pricing kernel ($-\ln m_{0,t}^j$) as a time-changed Levy process, $X_t^j$ ($j = US, JP$) with negative skewness. Then, \[ \ln S_t / S_0 = \ln m_{0,t}^{JP} - \ln m_{0,t}^{US} = X_{T_t}^{US} - X_{T_t}^{JP}. \]
  - Consistent and simultaneous modeling of all currency pairs.


- Reverse engineer the pricing kernel of US, UK, and Japan using currency options on dollar-yen, dollar-pound, and pound-yen.
Stochastic volatility/skew in currency options

![JPYUSD Implied Volatility](image1)

![GBPUSD Implied Volatility](image2)

![JPYUSD RR10 and SM10](image3)

![GBPUSD RR10 and SM10](image4)
Stochastic volatility/skew in SPX options
Tractable activity rate dynamics

- Most works use the square root process (*affine*):

  \[ dv_t = \kappa(\theta - v_t)dt + \sigma_v \sqrt{v_t}dW_t. \]

- Including jumps with constant (Eraker, Johannes, Polson (2003)) and proportional (Wu (2005)) arrival rates.
- Multi-dimensional extensions, with interactions (Duffie, Pan, Singleton (2000), Carr and Wu (2004)).

- **Quadratic** models: \( v_t \) is a quadratic function of an Ornstein-Uhlenbeck process (Santa-Clara and Yan (2005)), multivariate versions (Carr and Wu (2004)).

- **3/2** models (Heston, Lewis):

  \[ dv_t = \kappa v_t(\theta - v_t)dt + \sigma_v v_t^3 dW_t. \]

Much evidence favoring 3/2 over 1/2 in one-factor diffusion setting:

III. Model assembly
Model assembly

- Start with the risk-neutral ($Q$) process — That’s where tractability is needed the most dearly.
  - Identify the economic sources ($X^k_t$ for $k = 1, \cdots , K$)
  - Decide whether to apply separate time changes: $X^k_t \rightarrow X^{kT}_t$
  - Adjust to guarantee the martingale condition: $E^Q[S_t/S_0] = e^{(r-q)t}$.

$$\ln S_t/S_0 = (r - q)t + \sum_{k=1}^K \left( b^k X^{kT}_t - \varphi_{x^k} (b^k) T^k_t \right),$$

- $E^Q[e^{bX_t}] = e^{\varphi(b)t}$. Hence, $E^Q[e^{bX_t - \varphi(b)t}] = 1$, $E^Q[e^{bX^{T}_t - \varphi(b)T_t}] = 1$.

- Example: A CAPM model (with Levy return shocks and stochastic volatility):

$$\ln S^j_t/S^j_0 = (r - q)t + \left( \beta^j X^{mT}_t - \varphi_{x^m} (\beta^j) T^m_t \right) + \left( X^{i^jT}_t - \varphi_{x^j} (1) T^j_t \right).$$

Henry Mo:

- Estimate $\beta$ and market prices of return and volatility risk using index and single name options.
- Cross-sectional analysis of the estimates.
Market prices and statistics dynamics

- Since we can always use Euler approximation for model estimation, tractability requirement is not as strong for the statistical dynamics.

- We can specify pretty much any forms for the market prices subject to (i) technical conditions, (ii) economic sensibility, and (iii) identification concerns.

- Simple/parsimonious specification: Constant market prices of return and vol risks $(\gamma_k, \gamma_{kv})$

$$\mathcal{M}_t = e^{-rt} \prod_{k=1}^{K} \exp \left( -\gamma_k X^k T^k_t - \varphi_x (-\gamma_k) T^k_t - \gamma_{kv} X^{kv}_T - \varphi_{kv} (-\gamma_{kv}) T^k_t \right) \cdot \zeta,$$

- $\sigma W_t \rightarrow$ constant drift adjustment $\eta = \gamma \sigma^2$.

- Pure jump Lévy process $\rightarrow \pi^P(x) = e^{\gamma x} \pi^Q(x)$, drift adjustment:

  $\eta = \varphi^P_J(1) - \varphi^Q_J(1) = \varphi^Q_J(1 + \gamma) - \varphi^Q_J(\gamma) - \varphi^Q_J(1)$.

- Time change: instantaneous risk premium $(\eta v_t)$ proportional to the risk level $v_t$. 
Arbitrarily flexible market price of risks

\((\gamma_k, \gamma_{kv})\) are complicated functions of \((X^k_{T_t}, X^{kv}_{T_t})\) and other state variables

- The BS model \((\sigma W_t)\) with \(\gamma = \gamma_0 + \gamma_1 Z_t + \gamma_2 Z^2_t + \gamma_3 Z^3_t\), \(\mathbb{P}\)-dynamics:

\[
dS_t/S_t = (r - q + (\gamma_0 + \gamma_1 Z_t + \gamma_2 Z^2_t + \gamma_3 Z^3_t) \sigma^2) dt + \sigma dW_t.
\]

- \(Z_t\) can be some variables that predict asset returns (dividend yields, term spreads, default spread, etc).

- The Heston model with \(\gamma^v_t = \gamma_0/v_t + \gamma_1 + \gamma_2 v_t + \cdots + \gamma_k v^{k-1}_t\), and

\(\mathbb{Q}\)-dynamics: \(dv_t = (a - \kappa v_t) dt + \sigma_v \sqrt{v_t} dW^v_t\),

Then, its \(\mathbb{P}\)-dynamics

\[
dv_t = \left((a + \gamma_0 \sigma^2_v) - (\kappa - \gamma_1 \sigma^2_v) v_t + \gamma_2 \sigma^2_v v^2_t + \cdots + \gamma_k \sigma^2_v v^k_t\right) dt + \sigma_v \sqrt{v_t} dW^v_t,
\]

- Note: \(\gamma^v_t\) is the market price on \(\sigma_v \sqrt{v_t} dW^v_t\), not on \(dW^v_t\).

- Cheridito, Filipovic, Kimmel (2003), Pan& Singleton(2005): \(\gamma_k = 0\) for \(k \geq 2\).

- With \(\gamma_0 \neq 0\), risk premium approaches a finite amount when risk \((v_t)\) goes to zero. \(\rightarrow\) Economically sensible? even if it is mathematically ok (no arbitrage).
IV. Option pricing
To compute the time-0 price of a European option price with maturity at $t$, we first compute the Fourier transform of the log return $\ln S_t/S_0$. Then we compute option value via Fourier inversions.

The Fourier transform of a time-changed Lévy process:

$$\phi_Y(u) \equiv \mathbb{E}^Q \left[e^{iuX_T_t} \right] = \mathbb{E}^M \left[e^{-\psi_x(u)T_t} \right], \quad u \in D \subseteq \mathbb{C},$$

where the new measure $\mathbb{M}$ is defined by the exponential martingale:

$$\left. \frac{d\mathbb{M}}{d\mathbb{Q}} \right|_t = \exp \left( iuX_T_t + T_t\psi_x(u) \right).$$

- Tractability of the transform $\phi(u)$ depends on the tractability of (i) $\psi_x(u)$, and (ii) the Laplace transform of $T_t$ under $\mathbb{M}$.
- Tractable $\psi_x(u)$ comes from the Lévy specification: diffusion, compound Poisson, DPL, NIG,...
- Tractable Laplace comes from activity rate dynamics: affine, quadratic, 3/2.
- The two $(X, T_t)$ can be chosen separately as building blocks, for different purposes.
Fourier inversion for a cumulative distribution

Example: a European call: $C(k) = C(K, t)/S_0 = e^{-rt} E_0^Q [(e^{st} - e^{k})1_{st \geq k}].$

1. Treat $C(x) = C(k = -x) = e^{-rt} E_0^Q [(e^{st} - e^{-x})1_{-st \leq x}]$ analogous to a cumulative distribution.

   - The option transform:
     
     $$
     \chi_c(z) \equiv \int_{-\infty}^{\infty} e^{ikx} dC(x) = e^{-rt} \frac{\phi_s (-i - z)}{1 - iz}, \quad z \in \mathbb{R}.
     $$

   - The inversion is analogous to that for a cumulative distribution:
     
     $$
     C(x) = \frac{1}{2} \chi_c(0) + \frac{1}{2\pi} \int_{0}^{\infty} \frac{e^{ix} \chi_c(-z) - e^{-ix} \chi_c(z)}{iz} dz.
     $$
     
     with $\chi_c(0) = e^{-qt}.$

   - The literature often treats the call as a combination of two contingent claims and does the inversion separately. $C(x) = e^{-qt} Q_1(x) - e^{-rt} e^{-x} Q_2(x),$

   - Use quadrature methods for the numerical integration.
Fourier inversion for a probability density

II. Treat $C(k)$ analogous to a probability density function.

- The option transform:

$$
\chi_p(z) \equiv \int_{-\infty}^{\infty} e^{izk} C(k) dk = e^{-rt} \frac{\phi_s(z - i)}{(iz)(iz + 1)}
$$

with $z = z_r - iz_i$. We need $z_i \in \mathcal{D} \subseteq \mathbb{R}^+$ for the call option transform to be well defined.

- The inversion is analogous to that for a probability density:

$$
C(k) = \frac{1}{2} \int_{-iz_i - \infty}^{-iz_i + \infty} e^{-izk} \chi_p(z) dz = \frac{e^{-z_i k}}{\pi} \int_0^\infty e^{-iz_r k} \chi_p(z_r - iz_i) dz_r.
$$

- The numerical integration can be cast into an FFT to improve the computational speed. Obtain options across all strikes simultaneously.

- Use fractional FFT to separate the choice of strike grids from the integration grids (Chourdakis (2005)).
Concluding remarks

Modeling security returns with (time-changed) Lévy processes enjoys three key virtues:

- **Generality**: Lévy process can be made to capture any return innovation distribution; applying time changes can make this distribution vary stochastic over time.

- **Explicit economic mapping**: Each Lévy component captures shocks from one economic source. Time changes capture the relative variation of the intensities of these impacts.

- **Tractability**: Combining any tractable Lévy process (with tractable $\psi(u)$) with any tractable activity rate dynamics (with a tractable Laplace) generates a tractable Fourier transform for the time changed Lévy process. The two specifications are separate.

It is a nice place to start with for generating security return dynamics that are parsimonious, tractable, economically sensible, and statistically performing well.