Simple Robust Hedging with Nearby Contracts

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Abstract
This paper proposes a new hedging strategy based on approximate matching of contract characteristics instead of risk sensitivities. The strategy hedges an option with three options at different maturities and strikes by matching the option function expansion along maturity and strike rather than risk factors. Its hedging effectiveness varies with the maturity and strike distance between the target and the hedge options, but is robust to variations in the underlying risk dynamics. Simulation analysis under different risk environments and historical analysis on S&P 500 index options both show that a wide spectrum of strike-maturity combinations can outperform dynamic delta hedging.

Key words: characteristics matching, hedging, jumps, Monte Carlo, payoff matching, risk sensitivity matching, strike-maturity triangle, stochastic volatility, S&P 500 index options, Taylor expansion

JEL classification: G11, G13, G51

The concept of dynamic hedging has revolutionized the derivatives industry by drastically reducing the risk of derivative positions through frequent rebalancing of a few hedging instruments. Black and Scholes (1973) and Merton (1973) first developed the concept under a one-factor diffusion setting, but the underlying idea of matching risk sensitivities can be readily extended to multi-factor continuous risk dynamics and has been widely adopted in the industry for its simplicity and its wide applicability. One drawback, however, is that the hedge can deteriorate or even break down when the actual risk sensitivities deviate from what is assumed in the hedge, when unknown and hence unhedged risk sources show up,
and when the underlying risk sources can jump by a random amount. Miscalculating risk sensitivities or missing risk sources can happen because investors do not know the true risk dynamics and can only compute the risk sensitivities based on assumed dynamics and estimated coefficients. The presence of random jumps, on the other hand, is evidenced by various market crashes and other sudden large moves.

An alternative to matching risk sensitivities is matching payoffs, which is by design more robust because contracts with matching payoffs will behave similarly regardless of the underlying risk dynamics. With matched payoffs, investors have both values and risk sensitivities automatically matched at all times and under all scenarios, thus obviating the need to worry about mis-calculating risk sensitivities or the presence of random jumps. Breeden and Litzenberger (1978) pioneered this approach by showing that a path-independent payoff can be replicated using a portfolio of standard options maturing with the claim. This strategy is completely robust to model mis-specification, but the class of claims that this strategy can hedge is fairly narrow: Its hedge of a standard option reduces to a tautology and it cannot hedge options with different maturities. By assuming one-factor Markovian dynamics, Carr and Wu (2014) extend the strategy to hedge a long-term option with a continuum of shorter-term options. Carr and Chou (1997) propose static replications of barrier options using vanilla options under the Black and Scholes (1973) environment. These strategies are all designed in a case-by-case manner for a very particular set of contracts under specific market conditions. They cannot be readily generalized and hence have limited applicability.

In this paper, we propose a new option hedging strategy based on approximate matching of contract characteristics. We Taylor expand the option value function for both the target and the hedge options along the option expiry and strike price, and construct hedge portfolios to match the expansion terms with the target option. The approach combines the generality of the dynamic hedging approach with the robustness of the payoff matching approach. The Taylor expansion approach is analogous to the first-order risk sensitivity matching in dynamic hedging strategies and therefore represents a more generally applicable approach than the case-by-case perfect payoff replication literature. Meanwhile, contracts with matching characteristics generate matching payoffs and hence have matching price behaviors regardless of the underlying risk dynamics. Thus, the method is robust by design as its performance does not depend much on assumptions of the underlying risk dynamics.

Although expansions along contract characteristics are mathematically similar to expansions along risk factors, the two types of expansions generate drastically different economic implications. Both types of expansions can have higher-order expansion errors. For expansions along risk factors, the size of the expansion error is dictated by the size of the random shocks from the risk sources. As a result, the effectiveness of the dynamic hedging approach relies on the combination of frequent rebalancing and diffusive risk dynamics to keep the size of the random shocks small. Despite frequent rebalancing, dynamic hedge can still break down in the presence of large random shocks (i.e., jumps) or when the actual risk dynamics differ from what is assumed in the hedge construction. By contrast, for expansions along contract characteristics, the size of the expansion error depends mainly on the

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1 The idea has been further elaborated by Green and Jarrow (1987), Nachman (1988), and Carr and Madan (2001).
maturity and strike distance from the expansion point. Accordingly, hedging effectiveness of our approach relies on choosing appropriately spaced hedge contracts. The hedge is more effective when formed with nearby contracts, regardless of the underlying risk dynamics. Therefore, by expanding along contract characteristics instead of risk sources, we provide hedgers with more control of the hedging effectiveness through appropriate choices of hedge contracts while reducing the dependence on the underlying risk dynamics, which hedgers neither know with certainty nor have much control.

The idea of using nearby contracts to reduce hedging error is particularly important and is intuitively applicable even to dynamic hedging in practice. Only when the risk dynamics are purely diffusive, risk sensitivities are perfectly known, and rebalancing is continuous, does dynamic hedging performance become independent of instrument choice. In all practical situations, choosing nearby contracts with similar characteristics generate more robust hedges that can better endure large moves, unknown risk sources, and discrete rebalancing. Our strategy provides an explicit role to contract similarity in hedging instrument choice, and offers a formal justification for a common practice in the industry.

To put the different hedging approaches in perspective, we can think of dynamic hedging, or sensitivity matching, as defining the minimum requirement that removes known, diffusive risks for the next instant. Perfect payoff matching not only satisfies this minimum requirement, but also removes unknown and/or jump risks, at the expense of drastically reducing the available hedging choices. By approximately matching payoffs via expansions along contracts characteristics, our approach retains the general applicability of the dynamic hedging approach, while also taking advantage of the realization that a hedge is stable (with less need for rebalancing) and more robust (in the presence of unknown risk sources or randomly large shocks) when payoffs are better matched with nearby contracts.

Formally, our proposed strategy hedges a plain vanilla option with three other vanilla options on the same underlying security by matching maturity expansion to the first order and strike expansion to the second order, and by linking the butterfly spread (second derivative on strike) to the calendar spread (first derivative on maturity) via the local volatility definition of Dupire (1994). We show that the thus-formed hedge portfolio can be regarded as a finite difference approximation of Dupire’s forward partial differential equation (PDE). The finite difference approximation error is smaller and hence the hedge is more accurate when the maturity distance between the target and the hedge options is shorter and when the hedge option strikes are well spread around the target option strike.

We perform extensive Monte Carlo analysis on commonly used security price dynamics to gain insights on the optimal hedge strike spacing around the target option strike, and to verify the positive linkage between the size of the hedging errors and the target–hedge maturity distance. We also compare the performance of dynamic delta hedging with our three-option hedge under the different model environments. We consider both types of hedging over a one-month horizon, during which the delta hedging is rebalanced daily with the underlying futures whereas our three-option hedge portfolios are held static over the whole period. The transaction costs from the two approaches are largely comparable because the underlying security tends to be more liquid with narrower bid-ask spreads than the options but the dynamic hedge asks frequent trading of the underlying whereas our option portfolio are held static over the 1-month horizon. The exercise shows that the delta hedge performs
very well under a one-factor diffusion environment, but its performance deteriorates mark-
edly in the presence of stochastic volatility, and essentially breaks down in the presence of
large random jumps. By contrast, our static hedges generate superior and stable hedge per-
formances across all model environments, highlighting the inherent robustness of the con-
tract characteristics matching approach.

We also perform a historical analysis of S&P 500 index (SPX) options to compare the
actual performance of the two strategies under practical situations. The analysis shows that
from the traded option contracts, we can form many option hedge portfolios with different
maturity combinations that can significantly outperform the daily delta hedging strategy.
The outperformance is particularly strong and statistically significant when the average ma-
turity difference between the target option and the hedge options is within 6 months.

While theoretic developments often strive for perfect hedges under idealistic trading en-
vvironments and specific model assumptions, practical implementations must face transac-
tion costs and unknown risk dynamics as facts of life, and favor hedging strategies that
work reasonably well regardless of what happens in the market. Our new approach is par-
ticularly suited for practical implementation for its practical feasibility and generality in
hedge construction and for its robustness in performance.

In related literature, Renault and Touzi (1996) consider optimal hedging under a sto-
chastic volatility model. Hutchinson, Lo, and Poggio (1994) propose to estimate the hedg-
ing ratio empirically using a nonparametric approach based on historical data. He et al.
(2006) and Kennedy, Forsyth, and Vetzal (2009) set up a dynamic programming problem
in minimizing the hedging errors under jump-diffusion frameworks and in the presence of
transaction cost. Branger and Mahayni (2006, 2011) propose robust dynamic hedges in
pure diffusion models when the hedger knows only the range of the volatility levels but not
the exact volatility dynamics. For static payoff matching strategies, Balder and Mahayni
(2006) consider discretization strategies for the theoretical spanning relation in Carr and
Wu (2014) when the strikes of the hedging options are pre-specified and the underlying
price dynamics are unknown to the hedger.

The remainder of the paper is organized as follows. The next section defines the hedging
procedure, derives the optimal weights for the hedge portfolio, and illustrates its linkage to
finite difference schemes. Section 2 provides comprehensive simulation analysis to examine
optimal hedge option strike spacing around target options, to verify the hedge performance
dependence on target–hedge maturity distance, and to compare the hedge performance
with dynamic delta hedging under different model environments. Section 3 applies the
hedging strategy to a long history of SPX options and compares the performance to that of
a dynamic delta-hedging exercise with the underlying futures and daily rebalancing. Section
4 provides concluding remarks and directions for future research.

1 A New Framework for Hedging with Nearby Contracts

The idea of hedging with nearby contracts through characteristics matching is general. We
illustrate the idea with the example of hedging a vanilla option with three other vanilla op-
tions on the same underlying security. Plain vanilla option contracts are fully characterized
by their type (put or call), their expiry, and their strike prices. With no loss of generality,
we focus our discussions on call options. The results can be readily extended to puts via
put-call parity.
1.1 Assumptions and Notations
We use \( C(K, T) \) to denote the time-\( t \) value of a call option at strike \( K \) and expiry \( T \). To avoid notational cluttering, we assume zero rates and suppress the dependence of the option value on calendar time \( t \), the spot price \( S_t \), and other potential risk sources when no confusion shall occur. The representation highlights our focus on contract characteristics \((K, T)\) instead of risk sources.

To avoid the potential complication of limiting behaviors at extreme strikes and maturities, we limit our attention to options within a finite range of moneyness and expiration:

\[ j \ln \left( \frac{K}{S} \right) < K \text{ and } T < T, \]

for some fixed finite levels of \( K \) and \( T \). We further assume that at all strikes \( K \) and expiries \( T \) within this range, calendar spreads \( C_T(K, T) \equiv \frac{\partial C(K, T)}{\partial T} \) and butterfly spreads \( C_{KK}(K, T) \equiv \frac{\partial^2 C(K, T)}{\partial^2 K} \) are strictly positive, such that the Dupire (1994) local volatility is well-defined and strictly positive,

\[
\sigma^2(K, T) \equiv \frac{2C_T(K, T)}{K^2 C_{KK}(K, T)} > 0. \tag{1}
\]

Dupire derives the forward PDE in a one-factor diffusion setting, but the notion of local volatility in Equation (1) is well-defined under a much more general setting. Our proposed hedging strategy relies on a strictly positive local volatility estimate to link butterfly spreads to calendar spreads. Since in practice only a finite number of options are available at a discrete number of strikes and maturities, we perform interpolation and extrapolation on the observed option prices to evaluate the maturity and strike derivatives in generating the local volatility estimates.

We start at time \( t \) with a unit short position in a vanilla call option with strike \( K \) and expiry \( T \), and consider hedging this option position by using three call options at nearby strikes and maturities on the same underlying security.

In the absence of transaction cost, one can in principle form a hedge portfolio with more options to achieve better hedging performance. Practical concerns on transaction cost motivate us to constrain the hedge portfolio to a small number of options. In particular, we will show that three options are sufficient to achieve first-order characteristics matching between the target option and the hedge portfolio.

The three options can be at three different maturities and strikes. For our illustration, since there are often fewer maturities available than strikes, we limit our attention to a maturity-strike triangle formulation, where the three options in the hedge portfolio have three different strikes \( K_d < K_c < K_u \) but two different maturities, with the center strike \( K_c \) at one maturity \( T_c \) and the two outside strikes \( (K_d, K_u) \) at another maturity \( T_o \). It is also possible to choose the three options from one maturity with \( T_c = T_o \) as a special case. There is no particular restriction on the order of the three maturities \( T_c, T_o, T \), but one is more likely to choose more liquid, shorter-term options to hedge the possibly less liquid, longer-term option, that is, \( T_c, T_o < T \). Finally, it is natural to choose the hedge option strikes around the target option strike \( K_d < K_d < K_u \). A particularly simple and symmetric choice is to place the central strike at the target strike \( K_c = K \) and to set the two outer strikes equal-distance to the center \( K - K_d = K_u - K \).

\[ 2 \text{ A numerically more stable approach is to convert observed option prices into Black and Scholes (1973) implied volatilities, and compute the local volatility directly from the interpolated implied volatility surface, for example, Coleman, Li, and Verma (1998), Lee (2005), and Gatheral (2006).} \]
To characterize the distance of the hedge options to the target option, we introduce two standardized measures. One is a standardized measure of strike spacing around the target option strike,

\[ d_j \equiv \frac{K_j - K}{\sigma(K, T_o) K \sqrt{T - T_o}}, \quad j = d, c, u. \]

Intuitively, the standardized spacing measure \( d_j \) approximates the number of standard deviations that the security price needs to move from \((T_o, K_j)\) to \((T, K)\). While the hedge options can be at two different maturities, we use the maturity \( T_o \) as the reference maturity in this strike spacing definition. We also define a relative maturity spacing measure,

\[ \alpha \equiv \frac{T_o - T_c}{T - T_o}, \]

which measures the relative distance between the two maturities in the hedge portfolio to the distance between the target option maturity and the reference hedge maturity \( T_o \). \( \alpha = 0 \) when all three hedge options are placed on one maturity.

### 1.2 Matching Contract Characteristics Instead of Risk Sensitivities

The key idea for traditional dynamic hedging is to match risk sensitivities, that is, derivatives of the option valuation function against risk sources, between the target option and the hedge portfolio. When the underlying risk sources are diffusive and when the hedge portfolio can be frequently updated to maintain the risk sensitivity match, the hedge can be very effective. The drawback is that the hedge can be off when the assumed sensitivities deviate from reality, or when the underlying risk sources experience large random jumps.

Matching payoffs instead of risk sensitivities can generate more robust hedges that depend less on the underlying risk dynamics, but perfect payoff function matching can be either tautological (i.e., hedging the target with itself) or infeasible due to transaction costs. We propose to approximately match the target option’s payoff function using a small number of hedging instruments by matching the value function expansion along strike and expiry. With the same underlying security, the payoff of a vanilla call option contract is defined by its strike and expiry. Regardless of the underlying risk dynamics, contracts with the same strike and expiry shall behave the same. Through the expansion, we achieve approximate characteristics matching by matching the lower-order expansion terms with a small number of hedge options. The following proposition describes how to form a hedge portfolio that matches the target option in expansions along the two contract characteristics.

**Proposition 1:** To hedge the risk of a target option at \((K, T)\), we propose to use three hedge options to achieve an approximate payoff function matching by matching the option value function expansion against maturity to first order and against strike to second order. When we place the three options at three strikes \(K_d < K_c < K_u\) and two maturities \(T_o\) for \((K_d, K_u)\) and \(T_c\) for \(K_c\), the portfolio weights to achieve such matching are given by
which are well-defined as long as the matrix is nonsingular. When the three strikes are symmetrically placed around the target option strike with \( K_u - K = K - K_d = \Delta K \), the portfolio weights can be solved analytically as,

\[
\begin{align*}
  w_c &= \frac{d^2 - 1}{d^2 + \alpha}, \\
  w_d &= w_u = \frac{1}{2} (1 - w_c),
\end{align*}
\]

with \( d \) denoting the standardized strike distance,

\[
d = \frac{\Delta K}{\sigma(K, T_o) \sqrt{T - T_o}}. \tag{6}
\]

Proof. We perform Taylor expansion on both the target option and the hedge options at a common reference point \( (K, T_o) \). We expand the options along the maturity dimension to the first order and along the strike dimension to the second order,

\[
C(K, T) \approx C(K, T_o) + C_T(K, T_o)(T - T_o), \tag{7}
\]

\[
C(K_d, T_o) \approx C(K, T_o) + C_K(K, T_o)(K_d - K) + \frac{1}{2} C_{KK}(K, T_o)(K_d - K)^2, \tag{8}
\]

\[
C(K_u, T_o) \approx C(K, T_o) + C_K(K, T_o)(K_u - K) + \frac{1}{2} C_{KK}(K, T_o)(K_u - K)^2, \tag{9}
\]

\[
C(K_c, T_c) \approx C(K, T_o) + C_K(K, T_o)(K_c - K) + C_T(K, T_o)(T_c - T_o) + \frac{1}{2} C_{KK}(K, T_o)(K_c - K)^2. \tag{10}
\]

The expansions generate four terms \( C(K, T_o) \), \( C_K(K, T_o) \), \( C_{KK}(K, T_o) \), and \( C_T(K, T_o) \). We replace the butterfly spread \( C_{KK}(K, T_o) \) with the calendar spread \( C_T(K, T_o) \) via the local volatility definition in Equation (1). With this replacement, we can choose the portfolio weights \( (w_d, w_c, w_u) \) for the three options to match the coefficients on the three terms between the target option expansion and the hedge portfolio expansion.

Collecting terms and matching the coefficients on the reference option value \( C(K, T_o) \) between the target and the hedge portfolio, we have

\[
1 = w_d + w_u + w_c, \tag{11}
\]

which says that the hedge portfolio weights sum to one. Matching the coefficients on the strike spread \( C_K(K, T_o) \) leads to the following condition:

\[
0 = w_d(K_d - K) + w_u(K_u - K) + w_c(K_c - K), \tag{12}
\]
which says that deviations of the hedge option strikes from the target option strike should
average at zero. Finally, replacing \( C_{K_1}(K, T_o) \) with \( C_T(K, T_o) \) via the local volatility defini-
tion, matching the coefficients on \( C_T(K, T_o) \), and normalizing both sides by \( (T - T_o) \) lead
to the third condition on the portfolio weights,

\[
1 = \sum_j w_j \frac{(K_j - K)^2}{\sigma^2(K, T_o) K^2(T - T_o)} - w_c \frac{T_o - T_c}{T - T_o}, \quad j = d, u, c,
\]

which relates the weighted average strike difference squared between the target option and
the hedge portfolio to the maturity distance between these options.

Applying the standardized strike distance measure \( d \) defined in Equation (2) and the
relative maturity distance measure \( a \) in Equation (3), the second and third conditions in
Equations (12) and (13) can be rewritten as,

\[
0 = w_dd^2 + w_u d^2 + w_c d^2, \quad (14)
\]

\[
1 = w_dd^2 + w_u d^2 + w_c (d^2 - a). \quad (15)
\]

These conditions define the portfolio weights as purely a function of the standardized strike
distances \( d_j \) and maturity distance \( a \) between the target option and the hedge portfolio.
From the three conditions (11), (14), and (15), we can solve for the portfolio weights as in
(4) via a matrix inversion. In the special case of symmetric strike space, the solutions can be
written out analytically as in Equation (5).

By Taylor expanding the options along the strike and maturity dimension instead along
the risk sources, we construct a hedge portfolio that approximately matches the payoff be-
havior of the target option. The three conditions capture three major characteristics of the
payoff function: The first condition matches the scale between the target and the hedge
portfolio. The second condition asks that the hedge option strikes be placed around the tar-
get option strike so that the average strike of the hedge matches the target option strike.
The third condition explicitly recognizes the maturity difference between the target and the
hedge, and asks that the strikes of the hedge options spread more when the target option is
further away in maturity from the hedge options. In the special case when all three hedge
options lie on one maturity, the condition asks that the average squared percentage strike
distance between the hedge and the target be equal to the expected stock return variance
over the span of the maturity difference:

\[
\sum_j w_j \frac{(K_j - K)^2}{K^2} = \sigma^2(K, T_o) (T - T_o). \quad (16)
\]

Taken together, these conditions are constructed based on the strike and expiry place-
ments of the options and the expected security return variance over the expiry gap between
the target option and the hedge portfolio, but with no explicit dependence on the security
price level or calendar time.

Our proposed characteristics-matching approach has both striking similarities and sharp
contrasts with the sensitivity-matching dynamic hedging approach. Both approaches rely
on matching low-order expansion terms of the option value function. The expansion is
along risk sources for dynamic hedging but along contract characteristics for our new
The two types of expansions are mathematically similar, but they generate drastically different economic implications. The size of the error terms from expansions along the risk sources depends on the size of future movements of the risk sources. Thus, for the dynamic hedge to be effective, it is imperative that the risk sources move diffusively in the future and that the rebalancing is frequent so that the expansion error terms remain small. By contrast, the size of the error terms from expansions along the strike and maturity dimension depends on the strike and maturity differences between the target and hedge options. Unfortunately for dynamic hedge, investors cannot control the size of future movements in the risk sources; but fortunately for our new hedge, investors can proactively choose the maturity and strike placement for the hedge instruments to generate more stable hedges.

The different expansions also lead to different degrees of dependence on model dynamics. Since dynamic hedge relies on expansions along risk sources, its construction depends crucially on the assumed risk dynamics and its effectiveness depends crucially on the actual dynamics realization. For its construction, the number of hedging instruments is dictated by the number of assumed risk sources. For its effectiveness, the hedge generates zero hedging errors as long as the ex post dynamics are the same as the assumed diffusive dynamics and the hedger can rebalance the hedge continuously. Hedging errors occur when the assumed risk sensitivities differ from actual realization, when unknown/unhedged risk sources show up, or when the risk dynamics show random jumps. By contrast, since our strategy relies on expansions along contract characteristics, both its construction and effectiveness depend much more on contract structures than on risk dynamics. For its construction, if the target contact is a vanilla option at a certain strike and maturity, the hedge portfolio constitutes three vanilla options placed in strikes and maturities around the target option, the closer their placements are to the target option, the more effective the hedge is, regardless of how many underlying risk sources there are and how they behave.

Finally, because risk sensitivities can change quickly as the underlying security price or market conditions change, the dynamic hedge ratios can change quickly, too, necessitating frequent rebalancing. By contrast, since contract characteristics such as strike and expiry do not vary over time, once a hedge portfolio is constructed to approximately match the target option characteristics, they stay approximately matched over time. From Proposition 1, one can see that the three conditions and hence the portfolio weights are completely static as long as the underlying volatility does not change.

1.3 The Special Case of Placing the Three Hedge Options at One Maturity
When there are very few option maturities to choose from, hedgers can place all three hedge options at one maturity by setting $T_o = T_e$ and hence $x = 0$. In this case, we label the single maturity of the hedge options as $T_h$. With symmetric strike placement around the target option strike, the hedge portfolio weights become a simple function of the standardized strike distance measure $d$.

**Proposition 2:** When all three hedge options are placed at one maturity and symmetrically around the target option, the hedge portfolio weights become a simple function of the standardized strike spacing,
\[ w_c = 1 - \frac{1}{d^2}, \quad w_d = w_u = \frac{1}{2} (1 - w_c). \]  \hspace{1cm} (17)

When we approximate the target option with three strikes at one maturity \( T_h \), the approximation is analogous to a trinomial tree, and the weight on the center strike increases with the strike spacing. When the outside strikes are about one standard deviation away from the center \( d = 1 \), the center weight is zero and the trinomial tree degenerates into a binomial tree. When the strikes are spaced more than one standard deviation away, the weights on all three strikes become positive. For example, when the strike spacing is \( d = \sqrt{3}/2 \), the three strikes take on equal weight of \( \frac{1}{3} \) each. When the strikes are further apart at \( d = \sqrt{3} \), the center strike option takes on more weight at two-third, and the two outer strike options have a weight of one-sixth each.

Under a one-factor Markovian setting, Carr and Wu (2014) derive a static hedging strategy for a vanilla option \( C(K, T) \) using a continuum of options at a shorter maturity \( T_b < T \). Different from our approximations based on Taylor series expansions, their static hedge is an exact relation if (i) the underlying security price dynamics is known, (ii) the security price dynamics is one-factor Markovian, and (iii) a continuum of options are available at a shorter maturity to form the hedging portfolio. Given the practical constraints on contract availability and to minimize transaction costs, Carr and Wu propose a discrete-strike implementation procedure in which the strikes and portfolio weights are chosen based on a Gauss–Hermite quadrature approximation of the integral in the theoretical relation. When using three options to approximate the continuum hedge, the quadrature rule chooses symmetric strike spacing around the target strike with a standardized strike spacing of \( d \approx \sqrt{3} \) and with the portfolio weight for the center strike being two-third, the same as implied by Equation (17) in our Proposition 2. The fact that this particular choice of our hedge represents an approximation of a perfect payoff matching strategy under certain market conditions validates our intuitive argument that contracts with matched characteristics generate similar payoff behaviors. Furthermore, compared with their exact payoff matching theory, our characteristics-matching approach is much more generally applicable as it can be used to form a wide array of hedges based on different available strikes and maturities.

### 1.4 Analogy to Finite Difference Approximation Schemes

Proposition 1 provides the conditions to derive portfolio weights for a wide range of strike and maturity choices. Some hedge portfolios represent better approximations of the target option than others due to different sizes for the unmatched higher-order expansion terms. When multiple hedging instruments are available, it becomes an important research question to understand how to optimally choose the strike and maturity spacing of the hedge instruments to generate the most robust hedge portfolio. In this section, we show that there is an analogy between the hedge portfolio construction and finite different schemes for solving time-dependent PDEs. By showing this analogy, we hope that future research can lever the vast literature on finite difference schemes to better understand the approximation error behavior of the hedge portfolios.

We start with the special case of three strikes symmetrically placed at one maturity \( T_o = T \) as in Proposition 2. In this case, the target option is linked to the three hedge options by,
\[ C(K, T) \approx w_a C(K + \Delta K, T_o) + w_c C(K, T_o) + w_d C(K - \Delta K, T_o). \]  

(18)

Plug the portfolio weights into Equation (18) and rearrange, we have,

\[ \frac{C(K, T) - C(K, T_o)}{T - T_o} \approx \frac{1}{2} \sigma(K, T_o)^2 K^2 \frac{C(K + \Delta K, T_o) - 2C(K, T_o) + C(K - \Delta K, T_o)}{(\Delta K)^2}, \]  

(19)

with \( \Delta T = T - T_o \). Equation (19) represents a one-step, explicit finite difference scheme in approximating the Dupire (1994)’s forward PDE. Thus, this hedge portfolio can be viewed as a one-step marching for the Dupire’s equation using an explicit finite difference scheme. The stability of explicit finite difference schemes have been analyzed extensively in the literature. The approximation error increases with marching step size \( \Delta T \), in our case the maturity gap between the target option and the hedge portfolio. Furthermore, the stability condition for the explicit finite difference scheme (Courant, Friedrichs, and Lewy, 1928) requires that \( \Delta T \leq (\Delta K)^2/(\sigma(K, T_o)^2 K^2) \), or \( d^2 \geq 1 \). This condition guarantees that all three portfolio weights \( (w_a, w_c, w_d) \) are between 0 and 1.

When the central-strike hedge option is at a different maturity, say \( T_o < T_c < T \), we can regard the explicit finite difference scheme in Equation (19) as the first step, and then replace \( C(K, T_o) \) with \( C(K, T_c) \) in a second explicit finite difference step:

\[ \frac{C(K, T_c) - C(K, T_o)}{T_c - T_o} \approx \frac{1}{2} \sigma(K, T_o)^2 K^2 \frac{C(K + \Delta K, T_o) - 2C(K, T_o) + C(K - \Delta K, T_o)}{(\Delta K)^2}. \]  

(20)

The stability condition for this explicit step is that \( T_c - T_o \leq (\Delta K)^2/(\sigma(K, T_o)^2 K^2) \), which is automatically satisfied if \( T > T_c \) and the stability condition for the first step \( d^2 \geq 1 \) is satisfied. Combining the two finite-difference steps in Equations (19) and (20), we obtain an approximation of the target option with three options at \( (K + \Delta K, T_o) \), \( (K - \Delta K, T_o) \), and \( (K, T_c) \), with the portfolio weights given by

\[ w_c = \frac{d^2 - 1}{d^2 + 1}, \quad w_d = w_a = \frac{1}{2} (1 - w_c), \]  

(21)

the same as what we have derived in Equation (5). Therefore, when the hedge options are placed at two maturities, its construction is analogous to a two-step finite difference approximation of the forward PDE. The approximation accuracy is chiefly dictated by the size of the two approximation steps \( T - T_o \) and \( T_c - T_o \) and the strike placement needs to spread far enough and in proportion to the maturity gap to maintain hedge stability.

If the center-strike hedge option is at a shorter maturity, we can still start with the explicit scheme in Equation (19) as the first step, and then replace \( C(K, T_o) \) with \( C(K, T_c) \) in a second step:

\[ \frac{C(K, T_o) - C(K, T_c)}{T_o - T_c} \approx \frac{1}{2} \sigma(K, T_o)^2 K^2 \frac{C(K + \Delta K, T_o) - 2C(K, T_o) + C(K - \Delta K, T_o)}{(\Delta K)^2}. \]  

(22)

The solutions for the portfolio weights are the same, but this time the second step represents an implicit finite difference scheme, which is in general more stable.

---

In the special case of equal-distant maturity spacing $T - T_o = T_o - T_e = \Delta T$, we can combine the two steps into one step to generate the following finite difference representation,

$$
\frac{C(K, T) - C(K, T_c)}{2\Delta T} \approx \frac{1}{2} \sigma(K, T_o)^2 K^2 \frac{C(K + \Delta K, T_o) - (C(K, T) + C(K, T_e)) + C(K - \Delta K, T_o)}{(\Delta K)^2}.
$$

This scheme is referred to as the Du Fort-Frankel scheme in finite difference methods (Strikwerda 1989).

The analogy to finite difference schemes allows interested readers to leverage the vast literature in finite difference schemes in understanding the approximation error behaviors. It is well known that for both explicit and implicit finite difference schemes, the approximation errors are proportional to the time step. The further away in expiry the hedge options are from the target option, the larger the approximation error and hence the hedge error. For a given maturity distance between the target option and the hedge portfolio, numerical stability for the embedded explicit finite difference schemes requires that the strike spacing $\left(\frac{\Delta K}{2}\right)^2$ be large enough to guarantee positive weights on all three options in the hedge portfolio. The larger the maturity distance between the target option and the hedge portfolio, the further apart the hedge option strikes need to be placed to achieve stable hedging. Thus, when we talk about hedge with nearby contracts, the term “nearby” mainly refers to the target–hedge maturity gap. The hedge actually becomes less stable when the hedge strikes are placed too close to each other.

1.5 Target–Hedge Distance and Hedging Stabilities: A Numerical Illustration

Our hedge construction method highlights the importance of choosing hedging instruments that are close to the target contract in characteristics. Its linkage to finite difference schemes further shows that the key distance metric between the target and the hedge options in our setting is their maturity distance whereas the strike spacing of the hedge options should be wide enough relative to the maturity distance to achieve hedge stability. In this subsection, we use numerical examples to illustrate how the stability of the hedge depends on the maturity distance between the target and the hedge options.

For the illustration, we consider the sale of a one-year at-the-money call option, and compare three hedging choices: (i) delta hedge with the underlying, (ii) three 1-month options, and (iii) three 6-month options. The first hedge with the underlying can be regarded as having the largest distance between the target option and the hedge instrument, and the second hedge with 1-month options has a longer distance from the target option than the third hedge with 6-month options. For the second and third hedge with options, we place the three strikes symmetrically around the target option strike and fix the strike spacing at $d = 1.5$. Choosing other strike spacing generates similar illustration effects. We choose to use one single maturity for the option hedge portfolio to simplify the maturity distance discussion. Using two option maturities for the hedge portfolio can convey the same idea.

Figure 1 illustrates schematically how the value (Panel A) and delta (Panel B) of the hedged portfolio vary with the underlying security price movement. We generate the schematic values assuming that the underlying security price follows a geometric Brownian motion with zero interest rate and a constant volatility of 20%, and that the hedger knows the
true underlying risk dynamics in forming the hedges. We normalize the initial security price to \( S_0 = 100 \) so that the hedge portfolio value in panel A can be interpreted as percentages of the underlying security price. In each panel, the three lines denote the three hedges: delta hedge with the underlying in solid line, hedge with 1-month options in dashed line, and hedge with 6-month options in dash-dotted line. The circle in each panel denotes the target value the hedger intends to achieve.

In panel A on the hedged portfolio value variation, the hedger targets a hedged portfolio value of zero. Delta hedge can readily achieve this target at the initiation of the hedge with a combination of the delta of the underlying security and a cash position. However, the hedged portfolio value stays close to the target only under a narrow range of stock price movements. If the stock price can jump up or down by 50%, the hedged portfolio can lose over 15% of the underlying security value. By contrast, the two hedges with options do not exactly reach the target value of zero even at initiation because the hedges are formed based on low-order approximations. The approximations are closer to the target value with the 6-month options than with the 1-month options. When the underlying security price moves around, the hedged portfolio with 1-month options show some small variations, although much smaller than the delta-hedged portfolio. By comparison, the hedged portfolio with 6-month options stay very close to the target value of zero no matter how much the underlying security price moves, highlighting the increasing stability of the hedge when the hedge is formed with contracts closer to the target contract.

The delta variation of the hedged portfolio in Panel B tells a similar story. The delta hedge neutralizes the delta of the target to generate zero delta for the portfolio by design, but the delta of the portfolio starts to increase quickly as the security price drops and decrease as the security price increases, thus necessitating frequent rebalancing as the security price moves around. By contrast, the two characteristics-matched portfolios have very small delta exposure even though delta is not explicitly targeted in the portfolio construction, highlighting our argument that contracts with matched characteristics naturally match

![Figure 1. A schematic illustration of hedged portfolio value and delta variation with security price movements. The plots are created under the assumption of a geometric Brownian motion dynamics for the underlying security price with zero interest rate and a constant volatility of 20%. The hedge with stock matches delta. The hedges with options are constructed using our characteristics matching approach, where the three hedge options are placed at a single maturity and the three strikes are symmetrically placed around the target option strike with a standardized strike spacing of \( d = 1.5 \).](https://academic.oup.com/jfec/article-abstract/15/1/1/2548347)
price behaviors, including risk sensitivities. Furthermore, the delta of the hedged portfolio remain small across all ranges of the security price movements, obviating the need for frequent rebalancing. In particular, the delta neutralization is more complete and shows less variation with stock price movements for the hedge with 6-month options than with 1-month options, highlighting the role played by target–hedge maturity distance in hedging stability.

This schematic illustration shows that even under the framework of traditional delta hedging, there are many possible ways of neutralizing the delta of the portfolio. Our new hedge approach also neutralizes delta approximately even though the construction does not explicitly target delta neutralization. Within all possible choices of delta neutralization, choosing hedge contracts closer to the target contract is likely to bring more stability and robustness to the hedge both in terms of reducing the necessary rebalancing frequency and in terms of avoiding large losses in the case of unexpected large market movements. In this sense, the implications from our new approach can be applied to traditional dynamic hedging practices. In particular, even when the hedge construction method is based on neutralizing risk sensitivities, the hedger can consciously choose the nearby available contracts to make the hedge more robust and more stable. For example, to neutralize the vega (volatility exposure) of a 10-year option position, it is much safer to use a 9-year or 11-year option when available than to use a 1-month option. The different hedges may be equally effective in theory under an assumed one-factor diffusive stochastic volatility setting, but in practice the hedges with nearby options are much more likely to generate smaller hedging errors when actual realization deviates from model assumptions, for example, when volatility experiences sudden large jumps, or when short-term volatility moves in a different direction from long-term volatility.

1.6 General Applicability of Hedging with Nearby Contracts
The idea of matching characteristics in hedge portfolio construction is generally applicable to a wide range of securities, far beyond the particular vanilla option example that we use for illustration. While a full exploration of its application to other asset classes is left for future research, this section uses two examples to highlight its general applicability.

1.6.1 Hedging the interest-rate risk of zero-coupon bonds.
Traditionally, interest-rate risk is managed via estimating and matching interest-rate risk exposure of fixed income securities. In this example, we show how one can hedge the interest-rate risk of zero-coupon bonds by matching characteristics. Since the contract of a zero-coupon bond is purely defined by its maturity, we can simply expand the bond valuation function along the single characteristic dimension. Let $P(T)$ denote the value function of the target zero-coupon bond with expiry $T$, where we suppress the value dependence on risk sources and only show its dependence on contract characteristics. We form a hedge portfolio with two zero-coupon bonds at maturities $T_1$ and $T_2$ with weights $w_1$ and $w_2$, respectively. Expanding the two hedge instruments around the target maturity $T$ and matching the terms on $P(T)$ and $P_T(T)$ with that from the target bond, we obtain two simple conditions that determine the hedge portfolio weights:

$$1 = w_1 + w_2, \quad 0 = w_1(T_1 - T) + w_2(T_2 - T),$$

from which we have
The weights are positive when the target maturity is sandwiched by the two hedge maturities, $T_1 < T < T_2$. Given the first-order expansion, the weights represent a simple linear interpolation along the maturity dimension. The two hedge instruments are equally weighted when the maturities are equally spaced with $T - T_1 = T_2 - T = \Delta T$. Naturally, this linear approximation works well when performed within a narrow range, that is, when $\Delta T$ is small, regardless of how much interest rate can move or how many factors govern the interest rate term structure movements.

If the yield curve is driven by one diffusive interest-rate factor, one can hedge away the interest-rate risk completely via continuously rebalancing to maintain a duration of zero, regardless of which bond is used to neutralize the duration. In practice, however, no sensible practitioner regards neutralizing the duration of a 10-year bond with a 1-month bill as riskfree. Our approach does not guarantee complete risk removal, but highlights the fact that if one chooses two maturities sandwiching closely the target maturity, the portfolio is well hedged and the duration will remain small, regardless of whether there is one or many interest-rate factors, and regardless of whether the interest rate moves by a little or by a randomly large amount.

The simple construction method can be applied to both Treasuries and corporate bonds. For corporate bonds with significant default risk, the hedge remains effective during default events, as long as all bonds for the company share the same recovery rate.

1.6.2 Hedging barrier options.

While in theory delta hedging is applicable to both vanilla options and more exotic derivative structures such as barrier option, the delta hedge for barrier options can become unstable, especially when the security price becomes close to the barrier.

To apply our construction method to barrier options, we consider hedging a barrier option, say a call option with a down-and-out barrier, using three other down-and-out call options with the same barrier. Since the target and the hedge options share the same barrier, we can Taylor expand the barrier options just along the strike and maturity dimension, thus forming the hedge portfolio the same way as for vanilla options, except that we need to use the barrier option calendar spread and butterfly spread to define the volatility. The hedge is stable even when the security price moves close to the barrier. In particular, when the security price crosses the barrier, both target and hedge options drop to zero, keeping the hedge intact.

2 Monte Carlo Analysis of Commonly Specified Dynamics

We simulate several commonly specified security price dynamics and examine how different hedge portfolios perform under different dynamic environments. First, we analyze how to optimize the hedge option strike spacing for different maturity arrangements. Second, we verify the dependence of the hedging performance on the target–hedge maturity distance. Finally, we compare the hedging performance of our characteristic-matched hedge portfolio with dynamic delta hedging under different risk environments.
2.1 Data-Generating Processes


The time-series security price dynamics are governed by the following stochastic differential equations,

\[ \text{BS: } \frac{dS_t}{S_t} = \mu dt + \sigma dW_t, \]

\[ \text{MJ: } \frac{dS_t}{S_t} = \mu dt + \sigma dW_t + \int_{\mathbb{R}^0} (e^x - 1)(\nu(dx, dt) - \lambda n(x)dxdt), \]

\[ n(x) = \frac{1}{\sqrt{2\pi\nu_j}} \exp \left( -\frac{(x - \mu_j)^2}{2\nu_j} \right), \]

(26)

\[ \text{HV: } \frac{dS_t}{S_t} = \mu dt + \sqrt{\nu} dW_t, \]

\[ \text{HW: } \frac{dS_t}{S_t} = \mu dt + \sqrt{\nu} dW_t + \int_{\mathbb{R}^0} (\nu(dx, dt) - \nu_t \lambda_0 n(x)dxdt), \]

\[ d\nu_t = \kappa(\theta - \nu_t)dt + \omega \sqrt{\nu_t} dZ_t, \quad \mathbb{E}[dZ_t dW_t] = \rho dt, \]

where \( W_t \) denotes a standard Brownian motion in all four models. The MJ model incorporates a compound Poisson jump component, where \( \mathbb{R}^0 \) denotes the real line excluding zero, and \( \lambda n(x)dxdt \) is the compensator, with \( \lambda \) measuring the mean jump intensity, and \( n(x) \) denoting a normal probability density function capturing the jump size distribution in log return conditional on a jump occurring. Under the Heston (HV) model, \( Z_t \) denotes another standard Brownian motion that governs the randomness of the instantaneous variance rate. The two Brownian motions \( (Z_t, W_t) \) have an instantaneous correlation of \( \rho \). The HW model combines HV with MJ and allows the jump arrival rate to be proportional to the instantaneous variance rate, \( \nu_t = \nu_0 \nu_t \). The HW model is labeled as MJDSV3 in Huang and Wu (2004), who show that the model performs better in pricing SPX options than does a similar model with constant jump arrival rate proposed by Bates (1996) and Bakshi, Cao, and Chen (1997). Medvedev and Scaillet (2007) also find evidence that supports this dependence structure.

The four processes are carefully chosen for the analysis. The BS and MJ models serve as static pure diffusion and jump-diffusion benchmarks, respectively, whereas the HV and HW models allow stochastic volatility for the two benchmarks. Option prices under the BS model are computed using its analytical option pricing formula. Under the MJ model, option prices can be computed as a Poisson-probability weighted sum of the BS pricing formulae. For HV and HW, option prices are computed numerically through fast Fourier inversion of the return characteristic function.

To simulate the data-generating processes and price options on each simulated path, we need to choose appropriate values for the model parameters. To make the analysis comparable to our historical analysis on the SPX options in the next section, we set the parameter values to those calibrated to the SPX options market. Specifically, we perform daily calibration of the HV model and the HW model on SPX options from January 1996 to March 2009, and use the sample averages of the daily parameter estimates for the simulation analysis. The parameters for the BS model and the MJ model are adopted directly from the corresponding parameters from the HV and HW models, respectively, with the constant volatility level set to its long-run mean estimate. Table 1 reports the parameter values used
in our analysis. Estimating the HV model generates an average long-run mean volatility of \( \sqrt{\theta} = 22.77\% \), an average instantaneous volatility rate level of \( \sqrt{\nu_t} = 18.64\% \). The difference between the two implies an average upward sloping implied volatility term structure. The average mean-reversion coefficient is at \( \kappa = 3.7863 \). The average volatility of volatility coefficient estimate is quite large at \( \omega = 0.9095 \), which contributes to the curvature of the implied volatility smile. Finally, the average instantaneous correlation between return and return variance is strongly negative at \( \rho = -0.6824 \), consistent with the strongly negative skew observed in implied volatility plots against strike prices on SPX options.

By adding a jump component in the HW model, the average long-run mean volatility of the diffusion component becomes lower at \( \sqrt{\theta} = 18.69\% \) because the jump component also contributes to the total volatility level, which is at \( \sqrt{\theta(1 + \lambda_0(\mu_J^2 + \sigma_J^2))} = 22.44\% \), very close to the HV estimate. The average jump frequency is \( \lambda_0 \theta = 0.4995 \), about one jump every 2 years. Conditional on a jump occurring, the average jump size in return is \( \mu_J = -10.21\% \), with a standard deviation of \( \sigma_J = 14.32\% \). The large negative jump size contributes to short-term implied volatility skews in the SPX options, and the jump size uncertainty \( (\sigma_J) \) adds curvature to the skew. With the jump component, both the mean-reversion coefficient and the volatility of volatility coefficient average lower at \( \kappa = 1.8766 \) and \( \omega = 0.3811 \), respectively. The return-volatility correlation remains strongly negative at \( \rho = -0.7564 \).

The daily calibration on SPX options generates parameter estimates under the risk-neutral measure. To obtain the corresponding values for the statistical process, we assume zero risk premium by setting \( \mu = r - q \), and use the same set of parameters for both simulating the sample paths and pricing the options. During this sample period, the SPX started at 617.7, went over 1500 in year 2000 and 2007, but ended the sample at 822.92. The average ex-dividend return on the index over the sample period is 2.17%. The interest rates \( (r) \) and dividend yields \( (q) \) underlying the option contracts average at 4.17% and 2.58%, which we use as constants for simulating the data-generating processes and pricing the options.

### 2.2 Monte Carlo Procedures

To simulate the data-generating process, we first apply Ito’s lemma to derive the log security price process \( s_t = \ln S_t \). We simulate the time series of the log security price according to an Euler discretization of the respective data-generating process. For HV and HW models, Euler discretization can lead to negative realizations for the variance rate \( \nu_t \). We correct the situation using the full truncation method suggested by Lord, Koekkoek, and van Dijk (2010) and Medvedev and Scaillet (2010). When the simulated variance rate becomes negative, the full truncation method retains the negative value as the starting point for the next

<table>
<thead>
<tr>
<th>Model</th>
<th>( \sqrt{\theta} )</th>
<th>( \lambda_0 )</th>
<th>( \mu_J )</th>
<th>( \sigma_J )</th>
<th>( \sqrt{\nu_t} )</th>
<th>( K )</th>
<th>( \omega )</th>
<th>( \rho )</th>
</tr>
</thead>
<tbody>
<tr>
<td>BS</td>
<td>0.2277</td>
<td>—</td>
<td>—</td>
<td>—</td>
<td>—</td>
<td>—</td>
<td>—</td>
<td>—</td>
</tr>
<tr>
<td>MJ</td>
<td>0.1869</td>
<td>14.30</td>
<td>-0.1021</td>
<td>0.1432</td>
<td>—</td>
<td>—</td>
<td>—</td>
<td>—</td>
</tr>
<tr>
<td>HV</td>
<td>0.2277</td>
<td>—</td>
<td>—</td>
<td>—</td>
<td>0.1864</td>
<td>3.7863</td>
<td>0.9095</td>
<td>-0.6824</td>
</tr>
<tr>
<td>HW</td>
<td>0.1869</td>
<td>14.30</td>
<td>-0.1021</td>
<td>0.1432</td>
<td>0.1650</td>
<td>1.8766</td>
<td>0.3811</td>
<td>-0.7564</td>
</tr>
</tbody>
</table>
step of the variance rate propagation, but uses the truncated value $v_t^+ \equiv \max (0, v_t)$ in both
the security price propagation and the variance rate drift and volatility calculation. We also
price options using the truncated value $v_t^+$. The starting security price is normalized to $100
for all simulated sample paths. The starting values of the instantaneous variance rates under
the HV and HW models are fixed to the average values in Table 1.

We consider a hedging horizon of 1 month and simulate paths over this period. We as-
sume that there are 21 business days in a month. To be consistent with the historical ana-
lysis in the next section, we think of the simulation as starting on a Wednesday and ending
on a Thursday four weeks later, spanning a total of 21 weekdays and 29 actual days. The
security price moves according to the data-generating processes only on week days. For
each model, we simulate 1,000 sample paths and perform repeated hedging exercises on
these sample paths.

Figure 2 plots the 1,000 simulated sample paths for the security price under each of the
four model environments. The pure diffusive models BS and HV generate mostly small
price movements, whereas large discontinuous movements are apparent under MJ and
HW.

The HV and HW models also generate stochastic volatility. Figure 3 plots the corres-
ponding simulated sample paths for the instantaneous return volatility, $\sqrt{v_t^+}$, in percentage
points. Given the large volatility of volatility coefficient under the HV model, several

![Figure 2](https://academic.oup.com/jfec/article-abstract/15/1/1/2548347)

**Figure 2.** Simulated paths for the security price under different models. Lines represent the simulated
sample paths for the security price under different model environments.
volatility sample paths hit the lower bound of zero. By contrast, the estimated stochastic volatility dynamics under HW model are more well-behaved.

On each business day, we compute the relevant option prices based on the simulated realizations of the security price $S_t$ and the instantaneous variance rate, as well as the model dynamics. We monitor the hedging error based on the simulated security price and the option prices. The hedging error on each date $t$, $e_t$, is defined as the difference between the value of the hedge portfolio and the value of the target call option being hedged,

$$e_t = \sum_{j=1}^{3} w_j C_t(K_j, T_j) - C_t(K, T).$$

(27)

The portfolio weights for the three hedge options can vary over time if the local volatility estimate at the reference strike and expiry varies. Nevertheless, since the portfolios are constructed to approximate the time-invariant contract characteristics, we regard these variations as small and hold the initially constructed hedge portfolios static over the 1-month period. We investigate the hedging error of this static portfolio during the process.

We assume that option contracts are available at a finite number of strikes and maturities. We choose the target option and construct the hedging portfolio all from this pool of available option contracts. To compute the portfolio weights, we estimate the local volatility by interpolating the implied volatility surface constructed from the finite number of option observations.

At the start of each simulation, we assume that options are available at maturities of 1, 2, 3, 6, and 12 months, and that option strikes are centered around the normalized spot price of $100, and spaced at intervals of $1, $1.5, $2, $2.5, and $3 for the five maturities, respectively. The assumed strike spacing pattern matches the behavior of the SPX options market, where the strike spacing averages from $10 to $30 on an underlying index level of about $1,000.

We set the target option strike at $K = 100$, and consider three types of maturity-strike placements for the hedge portfolio: (A) Symmetric maturity-strike triangles with the center strike placed at a shorter maturity ($T_c < T_o < T$), (B) symmetric maturity-strike triangles with center strike placed at a longer maturity ($T_o < T_c < T$), and (C) symmetric strike

![Figure 3](https://academic.oup.com/jfec/article-abstract/15/1/1/2548347/151112548347) Simulated paths for the instantaneous volatility under stochastic volatility models. Lines represent the simulated sample paths for the instantaneous volatility, $\sqrt{\nu_t}$, under the HV and HW model, respectively.
placement on a single maturity \( (T_o = T_c < T) \). Within each type, we form 10 distinct target–hedge maturity combinations out of the five available maturities. For each maturity combination, we also have many flexible choices on the strike spacing. Through this extensive analysis, we examine the dependence of the hedging performance on maturity-strike placement patterns, target–hedge maturity distances, and strike spacing.

2.3 Optimal Strike Spacing Choice

For each maturity combination under each of the three maturity-strike placement types, we analyze the effect of strike spacing on the hedging performance. We start at strike spacing close to \( d = 1 \) and progressively move to the next available strike further away from the center strike \( K \). We perform the simulation for each strike spacing choice and record the hedging errors from 1,000 simulations, and we measure the hedging performance by comparing the terminal root mean squared hedging error (RMSE) at the end of the 1-month hedging exercise.

Figures 4–6 plot the terminal RMSE as a function of standardized strike spacing \( d \) for each maturity combination under each model environment. Each figure is for one maturity-strike placement type. Within each figure, each row represents one model environment, which contains 10 lines grouped into three panels, with each line representing one particular maturity combination. The legend shows the maturities in months in the sequence of \((T_c, T_o, T)\).

For each maturity combination, the RMSE plot shows a U-shaped pattern against the standardized strike spacing measure \( d \). Thus, when many strikes are available at each maturity, one can choose the appropriate strike spacing for the hedging portfolio to minimize the hedging error. We label the standardized strike spacing at the lowest RMSE as the optimal strike spacing, \( d^* \).

Figure 4 represents cases with the center strike placed at a shorter maturity \( (T_c < T_o < T) \). First, the RMSE tends to be higher when the target–hedge maturity gap is larger. For example, the three lines in the left panels all use 1- and 2-month options to hedge 3-, 6-, and 12-month options, respectively. With the hedging instruments fixed, the RMSE becomes larger for target options with longer maturities. The three lines in the middle panels all use 1-month option at the center strike and either 3- or 6-month options at the outer strikes to hedge 6- or 12-month options. The RMSE is the largest for the \((1,3,12)\) maturity combination (dash-dotted line), which has the largest target–hedge maturity gap. The four lines in the right panels use 2-, 3-, and 6-month options to hedge 6- and 12-month options, with the largest RMSE coming from the \((2,3,12)\)-maturity combination, the one with the largest maturity gap.

Second, the optimal standardized strike spacing \( d^* \) tends to vary with the relative maturity distance between the three hedge options. The standardized strike spacing measure \( d \) controls for the maturity gap between \( T \) and \( T_o \), but does not adjust for the center strike maturity. The distance between \( T_o \) and \( T_c \) relative to \((T - T_o)\) is captured by the relative maturity measure \( z \). The maturity combinations in each panel are ranked according to the relative maturity spacing measure \( z \) from high to low for the solid line, dashed line, dash-dotted line, and in the right panel, the dot-cross line. The optimal standardized strike spacing \( d^* \) tend to be smaller when the relative maturity spacing \( z \) is high. The minimum for the
solid lines almost always occurs at lower $d$ than that for the dash-dotted or dot-crossed line within the same panel. This pattern is particularly clear under the BS and MJ environments.

Figure 5 represents the cases with the center strike placed at a longer maturity ($T_c < T_o < T$). The allocation of each line corresponds to that in Figure 4, except with a switch between $T_c$ and $T_o$. As a result of the switch, the relative maturity spacing measures $\alpha$ are all negative, and the ranking from solid to dotted lines is from low (more negative) to high (less negative) $\alpha$. The ranking of the optimal strike spacing across different lines also switches, with the solid lines (with more negative $\alpha$) showing wider optimal strike spacing and the dotted lines (with less negative $\alpha$) showing narrower optimal strike spacing. This ranking pattern stays reasonably consistent across all four models and reveals a negative relation between $\alpha$ and $d^*_c$.
longer maturity. Also universal is the observation that the RMSE, at the optimal striking, is almost always larger when the target–hedge maturity gap is wider.

Figure 6 represents the cases with all three strikes in the hedge placed at the same maturity. What remains clear is the observation that the RMSE tends to be larger for larger target–hedge maturity gaps. With all three hedge options on one maturity and hence \( z = 0 \), it seems that optimal standardized strike spacing \( d^* \) becomes wider when the target–hedge maturity gap becomes narrower.

To quantify the observed dependence of the optimal standardized strike spacing on the relative maturity spacing, we aggregate the results from all 30 maturity combinations under each model environment and estimate the following relation:
where $\alpha$ captures the relative spacing between the two hedge maturities and $(T_0/T)$ captures the relative spacing between the target and the hedge options. The regression results are summarized in Table 2. The regressions explain 77–96% of the variation. Under all four model environments, the dependence of the optimal strike spacing on the two explanatory variables is similar. The optimal standardized strike spacing declines with increasing relative maturity spacing between the hedge options ($\alpha$), and it also declines with increasing relative maturity spacing between the hedge and the target options $(T_0/T)$. As the ratio becomes smaller and hence the distance becomes larger, the optimal strike spacing becomes smaller. These estimated relations provide a general guidance on the strike spacing choice in practical applications.
2.4 Hedging Performance Dependence on Target–Hedge Maturity Distance

By analogy to the finite difference schemes, the size of the hedging errors increases with the maturity distance between the target option and the hedge options because this maturity gap essentially determines the time step of the finite difference approximation. The numerical analysis example in Section 1.5 also shows that both the value and the delta of the hedged portfolio become more stable when the target–hedge maturity gap is narrower. To gain a more quantitative understanding on how the hedging performance varies with the target–hedge maturity gap, Figure 7 plots the terminal RMSE against the maturity distance \( \frac{T}{C_0} \frac{T_h}{C_0} \) between the target option and the hedge options, with \( T_h = w_c T_c + \left( \frac{1}{C_0} - w_c \right) T_o \) denoting the weighted average maturity of the hedge portfolio. The plots show that as the target option maturity becomes further away from the hedge options, the RMSE increases. The solid line represents a regression fitting of the increasing relation, which highlights the virtue of hedging with “nearby” contracts.

Figure 7 differentiates the three different types of maturity-strike combination patterns using different markers, with circles denoting maturity-strike triangles with \( T_c < T_o < T \), diamonds denoting maturity-strike triangles with \( T_o < T_c < T \), and squares denoting three strikes at one maturity \( T_o = T_c < T \). Although the hedging performance shows clear dependence on the maturity distance \( T - T_h \), the performances do not show large difference across the three patterns. Closer inspection shows more squares above the fitted line and more diamonds below the fitted line. Of the three types, the squares (three strikes on one maturity) involve one finite difference step in approximating the forward PDE, whereas the other two types involve two marching steps, thus potentially reducing the approximation error. The performance differences between the diamonds \( (T_o < T_c) \) and the circles \( (T_c < T_o) \) are potentially related to how the approximation errors in the two steps cancel or aggregate with each other.

2.5 Performance Comparison with Dynamic Delta Hedge

To gauge the relative effectiveness of our new hedge construction method, we compare the performance of our three-option static hedges to dynamic delta hedging with the underlying

<table>
<thead>
<tr>
<th>Model</th>
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<th>( b )</th>
<th>( c )</th>
<th>( R^2 )</th>
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<tr>
<td>BS</td>
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<td>-0.639</td>
<td>0.403</td>
<td>0.917</td>
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<tr>
<td>MJ</td>
<td>1.250</td>
<td>-0.650</td>
<td>0.888</td>
<td>0.957</td>
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<tr>
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<td>-1.246</td>
<td>2.222</td>
<td>0.768</td>
</tr>
<tr>
<td>HW</td>
<td>1.102</td>
<td>-1.059</td>
<td>1.511</td>
<td>0.851</td>
</tr>
</tbody>
</table>

Notes: Entries report results from the following regression

\[
d^* = a + b \alpha + c(T_o/T) + e,
\]

where the optimal strike spacing \( d^* \) is related to the relative maturity spacing between the hedge options \( \alpha \) and the relative maturity spacing between the hedge and target options \( (T_o/T) \) under each model environment. In parentheses are standard errors of the parameter estimates. The last column reports the \( R^2 \) of the regressions.

Table 2. Relate optimal strike spacing to relative maturity spacing among hedge and target options

\[\text{Model} \quad \text{abc} \quad \text{R}^2\]
\[\begin{array}{cccc}
\text{BS} & 1.381 (0.027) & -0.639 (0.039) & 0.403 (0.082) & 0.917 \\
\text{MJ} & 1.250 (0.018) & -0.650 (0.025) & 0.888 (0.054) & 0.957 \\
\text{HV} & 1.094 (0.091) & -1.246 (0.132) & 2.222 (0.277) & 0.768 \\
\text{HW} & 1.102 (0.057) & -1.059 (0.082) & 1.511 (0.173) & 0.851 \\
\end{array}\]
futures and daily rebalancing. Daily delta hedging represents the common practice in the industry. Following each simulated sample path, we compute the BS delta of the option at its current implied volatility level at each date and rebalance the futures position accordingly. The hedging error at each date $t$, $e_t$, is computed as,

$$e_t = B_{t-1}e^b h + \Delta_{t-1}(F_t - F_{t-1}) - C(S_t, t; K, T),$$  

(29)

where $\Delta_t$ denotes the delta of the target call option with respect to the futures price at time $t$, $h$ denotes the daily time interval between stock trades, and $B_t$ denotes the time-$t$ balance in the money market account, which includes the receipts from selling the target call option, less the cost of initiating and changing the hedge portfolio.

In principle, one can compute the delta based on some other assumed dynamics. Since investors do not know the true underlying risk dynamics, they must first assume some type of dynamics and then estimate the assumed dynamics to match the observed option price behaviors. Unfortunately, different types of dynamics can generate similar option price behaviors, but can have quite different implications for the hedge ratios. Therefore, despite advancements in option pricing models, the common industry practice remains to use the

---

**Figure 7.** Dependence of hedging performance on target/hedge maturity difference. $T$ denotes the target maturity and $T_h = w_c T_c + (1 - w_c) T_o$ denotes the weighted average hedge option maturity. Circles represent maturity-strike triangles with $T_o < T_c < T$. Diamonds represent maturity-strike triangles with $T_o < T_c < T$. Squares represent the hedge portfolios of three strikes at one maturity $T_c = T_o < T$. The solid line represents a linear regression fit. Each panel is for one underlying model.
BS formula to compute the delta at the observed implied volatility level of the option, for both simplicity and robustness. In the academic literature, most hedging ratio enhancements are proposed under some assumed model environments and thus do not satisfy the practical robustness requirement. Some studies, for example, Hutchinson, Lo, and Poggio (1994), propose to estimate the hedging ratio empirically, without assuming any model, by using a nonparametric approach based on historical data. One must however distinguish between robust methods, which perform well under most risk environments, and nonparametric or model-free approaches, which do not depend on model assumptions in their derivation or estimation, but do not guarantee robustness in any sense. Quite the opposite, many nonparametric approaches suffer in-sample overfitting and out-of-sample degeneration, and are therefore anything but robust.

Table 3 reports the terminal RMSE from daily delta hedging on the four different target options under each of the four different model environments, each compared with the summary performance from the corresponding three-option triangle hedges. For target options at 2 month maturity, we have formed one static portfolio with three 1-month options. For target options at 3-, 6-, and 12-month maturities, we have formed 4, 9, and 16 different hedge portfolios with different maturity combinations, respectively. Table 3 reports the minimum, mean, and maximum RMSE from these different combinations to compare with the RMSE from the daily delta hedge.

The delta hedge works remarkably well under the BS environment, with RMSE ranging from 0.075 to 0.203. Our three-option static hedges perform equally well. Hedging the 2-

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<td>Min  Mean Max</td>
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<th>MJ</th>
<th>Delta Triangles</th>
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</thead>
<tbody>
<tr>
<td>Target</td>
<td>Min  Mean Max</td>
<td>Min  Mean Max</td>
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</tr>
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<td>0.920 0.080 0.080</td>
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<tr>
<td>3</td>
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<tr>
<td>12</td>
<td>0.598 0.050 0.139</td>
<td>0.598 0.050 0.139</td>
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</tbody>
</table>

Notes: Under each model environment, for each target option, entries report the RMSE at the end of the 1-month hedging exercise from the dynamic daily delta hedging strategy with the underlying futures, and compare it with the minimum, mean, and maximum RMSE from different strike-maturity triangle hedges. To hedge the 2-, 3-, 6-, and 12-month target option, we have constructed 1, 4, 9, and 16 different strike-maturity triangle hedges, respectively.
month option with three 1-month options generates a very small RMSE of 0.045. The four portfolios for hedging the 3-month option generate RMSE from 0.012 to 0.106, all smaller than the dynamic delta hedging RMSE at 0.159. At 6- and 12-month target option maturity, some of the static hedges generate smaller RMSE than the delta hedge while others generate somewhat larger hedging errors. Overall, the performances from the two approaches are comparable.

When the underlying security price dynamics includes random jumps as in the MJ environment, the performance from the dynamic delta hedge deteriorates sharply. Compared with the BS environment, the RMSE estimates become several folds larger, from 0.920 for 2-month options and 0.598 for 12-month options. By contrast, our three-option static hedges perform just as well as under the BS model environment, with the mean RMSE for the four target options ranging from 0.080 to 0.139. The largest RMSE at 0.268 remains a modest number. Therefore, even the worst-performing static hedge performs drastically better than the dynamic delta hedge.

Introducing stochastic volatility, as is the case under the HV model environment, also sharply worsens the performance of the dynamic delta hedge, which generates RMSEs from 0.686 to 0.952. Stochastic volatility also reduces the stability of the static option hedges. The mean RMSEs from the static option hedges for the four target options become higher than under the BS and MJ environments, ranging from 0.228 to 0.407. Nevertheless, they are still less than half of RMSE estimates from the corresponding daily delta hedge.

Finally, the HW model environment includes both random jumps and stochastic volatility, inducing the worst performance from the daily delta hedging exercise. The RMSE estimates range from 0.861 to 1.076. By comparison, our static option hedges perform better on average than the HV case because the estimated volatility process is more stable under the HW environment. The mean RMSEs for the four target options range from 0.235 to 0.357, only fractions of the corresponding RMSE estimates from the dynamic delta hedge.

Comparing the hedging performance across different target options under each model environment, we find that under all model environments, the dynamic delta hedge works better for longer-dated options than for shorter-dated options. The larger gamma for short-dated options is potentially the source of larger delta hedging errors in presence of large price movements. By contrast, the largest hedging errors from the static option hedges tend to come from the longer-dated options. The source of the larger error, in this case, is the maturity distance between the target option and the hedge options. When hedging 2-month options, we use 1-month options with a target–hedge maturity gap of 1 month. When hedging 12-month options, we can choose hedge options with maturities from one month to six months. The maturity gap can therefore be much larger, leading to larger hedging errors.

To visualize the performance differences under different model environments, Figure 8 plots the simulated sample paths of the hedging errors on the 12-month option hedged with (i) daily delta hedging with the underlying futures and (ii) the maturity-strike triangle with the center strike at 2-month maturity and the outer strikes at 1-month maturity. The delta hedging of the 12-month option generates the best delta-hedging performance among the four target options. On the other hand, the chosen triangle is the farthest away from the target option in terms of maturity distance. Thus, we are comparing the best scenario from the delta hedging with the worst choice from the static option hedges.

The hedging errors under the BS model environment are very small for both strategies. The delta hedge generates a terminal RMSE of $0.08 whereas the terminal RMSE from the
The delta hedge error distribution is negatively skewed, with a skewness estimate of –0.63; whereas the options hedge generates positive skewness in the terminal hedging error distribution at 1.74. For both strategies, the hedging errors are quite small. The initial value of the target call option is $9.55 under the BS model. An RMSE of 0.08–0.22 is only 0.8–2.3% of the sales receipt of the target option. The maximum loss from the delta hedge is $0.33 and that from the triangle is merely $0.18. Both strategies perform well under the BS environment.

Under the MJ environment, whenever the underlying security price experiences a large jump of either direction, the delta-hedged portfolio experience a large negative error. The terminal hedging error distribution becomes strongly negatively skewed, with a skewness estimate of –11.04. The terminal RMSE is 0.60, and the maximum hedging loss is as high

<table>
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<tr>
<th>Model</th>
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<tr>
<td>HW</td>
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</table>

**Figure 8.** Comparing the simulated sample paths for the hedging errors. Lines represent the simulated sample paths for the hedging errors of a 12-month call option. The left panels are from daily delta hedging with the underlying futures. The right panels are from static hedging with the maturity-strike triangle with outer strikes at 1 month and center strike at 2 months.

triangle hedge is larger at $0.22. The delta hedge error distribution is negatively skewed, with a skewness estimate of –0.63; whereas the options hedge generates positive skewness in the terminal hedging error distribution at 1.74. For both strategies, the hedging errors are quite small. The initial value of the target call option is $9.55 under the BS model. An RMSE of 0.08–0.22 is only 0.8–2.3% of the sales receipt of the target option. The maximum loss from the delta hedge is $0.33 and that from the triangle is merely $0.18. Both strategies perform well under the BS environment.

Under the MJ environment, whenever the underlying security price experiences a large jump of either direction, the delta-hedged portfolio experience a large negative error. The terminal hedging error distribution becomes strongly negatively skewed, with a skewness estimate of –11.04. The terminal RMSE is 0.60, and the maximum hedging loss is as high
as $9.13, close to the sales revenue from the target option. Thus, the delta hedge essentially breaks down in the presence of large jumps. By contrast, our static options portfolio hedges both small and large price movements well. The maximum loss from the triangle hedge is $0.25 and the RMSE is only $0.17.

Under the HV environment, delta hedging generates a terminal RMSE of 0.69 whereas the options hedge generates a RMSE of 0.25. Therefore, the presence of stochastic volatility also strongly deteriorates the delta hedge performance, but it only leads to mild deterioration on the hedging performance of the maturity-strike triangle.

Under the HW environment, which includes both random jumps and stochastic volatility, it is the large jump that generates the largest hedging errors for delta hedging. In this case, the terminal delta-hedging loss can be as high as $9.91, more than the sales revenue from the target option. The terminal hedging error distribution is strongly negatively skewed, with a skewness estimate of $-4.38$ and an RMSE of 0.86. By contrast, the terminal hedging error distribution from the options hedge is relatively symmetric, with a skewness estimate of merely $-0.19$, and the RMSE remains small at 0.40.

These simulation exercises show that our proposed three-option static hedge dominates the daily rebalancing delta hedging in terms of the RMSEs. Most important, the dynamic hedge works well under the BS model environment, but its performance declines markedly in the presence of stochastic volatilities, and it virtually breaks down in the presence of large random jumps. By contrast, our three-option static portfolio generates stable hedging performance under all simulated environments, and thus represents a more robust alternative.

### 2.6 Transaction Cost Considerations

In computing the hedging errors from both types of strategies, we take the simulated prices without adjusting for transaction costs such as bid-ask spreads and market impacts. The dynamic delta hedge involves frequent trading of the underlying whereas our static hedge requires putting on static positions on three options at shorter maturities and closing the positions (or letting the options expire) 1 month later. In general, trading the underlying security costs much less than trading its options, but the updating frequency difference compensates for the cost per transaction difference. Take SPY, the ETF on the SPX, as an example. The market for SPY is very liquid and deep. The bid-ask spread is merely a penny at normal market conditions. By comparison, the most liquid SPY options, usually at short maturities and with strikes close to the spot level, tend to have bid-ask spreads of two-to-three cents. For less liquid contracts at longer maturities and strikes far away from the spot level, the spreads can increase to ten cents or more. Similar transaction cost comparisons also hold for the index futures and its options. Thus, per each trade, the cost for the underlying tends to be much lower than that for its options, but dynamic delta rebalancing requires trading the underlying every day whereas our static hedge only requires putting on the initial positions and holding them statically until the expiry of the shorter-dated options. The overall transaction costs from the two approaches become similar under normal market conditions.

There are other types of costs to consider in setting up hedging strategies such as monitoring costs and uncertainties regarding future transactions. For dynamic hedging strategies based on risk sensitivity matching, updating to rebalance the sensitivity match is the most dearly needed when the market experiences abnormally large movements, but these are
also times when efficient transactions become the most difficult to achieve. Thus, dynamic hedging experiences the most difficulty exactly when hedging is the most needed. When the market experiences large abnormal movements, not only the hedging errors from first-order sensitivity matching become large, but also rebalancing becomes difficult and costly. By contrast, a portfolio with matched characteristics stay matched regardless of market conditions. Therefore, although the overall transaction costs for the two strategies are largely comparable at normal times, they can show different characteristics, especially at times of market stress.

3 A Historical Hedging Exercise on SPX Options

This section investigates the historical hedging performance in hedging the sale of SPX options. We obtain from OptionMetrics data on SPX options from January 1996 to March 2009. These options are standard European options on the cash index and are listed at the Chicago Board of Options Exchange. The data set includes, among other information, the closing quotes on each options contract. The hedging exercises are based on the mid quotes of the options.

The design of the historical analysis parallels the simulation exercise in the previous section. Over the historical sample period, we identify 158 starting dates from January 17, 1996, to February 18, 2009, when there are options expiring exactly 30 days after. Since the SPX options expire on the Saturday following the third Friday and the terminal payoff is computed based on the opening price on that Friday morning, trades and quotes on the expiring options effectively stop on the preceding Thursday, and our chosen starting dates in each month all fall on a Wednesday. We start our hedging exercises on each of the 158 starting dates and monitor the hedging performance for the next 30 days. For each strategy, we compute the summary statistics of the hedging errors based on the 158 repeated exercises. The hedging errors from all exercises are normalized to be in percentages of the index level at the start of the exercise.

At each starting date, options are always available at 1-month maturity (31 days) by design. Two-month options are also available for all starting dates, but the maturity availability after the 2-month maturity varies across starting dates. For our hedging exercise, we classify options into four maturity groups: (i) 1-month options (31 days), (ii) 2-month options (59 or 66 days), (iii) options with maturities 3–5 months (87–157 days), and (iv) options with maturities around 1 year (276–402 days). For convenience, we refer to the latter two groups as 4-month and 12-month options, respectively. Based on the four maturities groups, we can form 14 target–hedge portfolio maturity combinations, with four satisfying $T_o < T_c < T$, four satisfying $T_c < T_o < T$, and the remaining six satisfying $T_c = T_o < T$.

In each of the 14 combinations, we choose the target option strike close to the spot level. To choose the strike spacing for the hedging portfolio, we use the regression results in Table 2 under the HW model to estimate the optimal strike spacing $d^*$ as a function of $\alpha$ and $T_o/T$. Choosing $d^*$ based on the simulation results from the other three models generates similar results. Given $d^*$, we compute $\Delta K$ and the portfolio weight based on the local volatility estimate $\sigma(K, T_o)$ and the maturity placement $\alpha$. Then, we choose the three
available strikes for the hedging portfolio that are closest to the estimated optimal strike spacing.

Figure 9 plots the terminal RMSE estimates against the target–hedge maturity distance \((T - T_h)\). Similar to the simulation results, the RMSE estimates increase with the maturity distance. The RMSE on SPX options are somewhat larger than the four simulated cases in Figure 7, potentially due to constraints on strike availability and the possibility that the SPX index dynamics are more complicated than those simulated. For comparison, we also perform the delta hedging with the underlying futures with daily rebalancing. The RMSEs on 2-, 4-, and 12-month options are 0.63, 0.63, and 0.66, respectively. Of the 14 maturity combinations for our three-option strategy, only three generate root mean squared errors larger than 0.63. The target–hedge maturity distances for the three hedges are among the widest, all greater than 8 months.

To gauge the statistical significance of the hedging performance differences between the two types of strategies, we compute the squared hedging error difference on each date, \(\delta_t = e_{D,t}^2 - e_{O,t}^2\), where \(e_{D,t}\) denotes the hedging error of the dynamic delta strategy on date \(t\), and \(e_{O,t}\) denotes the hedging error on the same date and on the same target option from one of our three-option hedge portfolios. We follow both strategies for 29 actual days, from the starting date to the Thursday of the fourth following week, the last day of trading for the 1-month options used in the static hedge. We have 14 static hedges that vary in the target option and/or the hedging options maturity. For each static hedge and on each date, we compute the squared hedging error difference with the corresponding dynamic delta

![Figure 9](https://academic.oup.com/jfec/article-abstract/15/1/1/2548347/151/2548347.png)
hedging on the same target option. We then construct the Diebold and Mariano (1995) (DM) $t$-statistics on the squared hedging error difference over the 158 sample exercises,

$$DM = \frac{\delta}{s_\delta},$$

where $\delta$ denotes the sample mean of the difference, $s_\delta$ denotes the standard error estimate. Under the null hypothesis that the two strategies have equal finite-sample hedging accuracy, Clark and McCracken (2012) find that the thus-computed Diebold and Mariano (1995) test statistic can be compared with standard normal critical values. A strongly positive statistic indicates that our hedge portfolio significantly outperforms the dynamic delta hedge on the same target option.

Figure 10 plots the DM-statistics across different days of the hedging exercise from start to finish. Since the performance of our hedge strategy depends on the target-maturity distance, we separate the 14 hedges into two broad groups. One group has target-maturity difference smaller than 6 months ($T - T_h < 0.5$), denoted in the graph with circles. The rest with larger target–hedge maturity gaps ($T - T_h \geq 0.5$) are denoted with diamonds. Virtually all of the maturity combinations with $T - T_h < 0.5$ perform significantly better than the corresponding delta hedge except at the very first few days of hedging. Most of the hedges with $T - T_h \geq 0.5$ also outperform the delta hedge, but the outperformance is less statistically significant.

![Figure 10](https://academic.oup.com/jfec/article-abstract/15/1/1/2548347)
4 Concluding Remarks

Traditional dynamic hedging is constructed by matching risk sensitivities between the target option and the hedge portfolio. The approach is applicable in a wide range of situations, but its effectiveness can suffer when the actual risk sensitivities deviate from the assumptions and when the underlying risk sources experience large unexpected jumps. This paper proposes a new hedging strategy that is more stable over time (and hence requires less updating) and is more robust with respect to variations in the underlying risk dynamics. The strategy uses three options to form a hedge portfolio that, instead of matching risk sensitivities, approximates the payoff behavior of the target option by matching Taylor expansion terms first order in option maturity and second order in option strikes. Option contract payoffs are dictated by their two contract characteristics: strike and expiry. Matching expansions along these two characteristics lead to approximate matching of payoff behaviors, regardless of the underlying risk dynamics. By analogy to finite difference schemes, we show that the approximation generates small errors when the maturity distance between the target option and the hedge options is small, and when the strikes of the three hedge options are well spread to satisfy stability conditions. Through extensive simulation exercise, we show how one can choose appropriate strike spacing between the three hedge options to minimize the hedge error, and we confirm how the RMSE depends directly on the maturity distance between the target option and the hedge portfolio.

When we compare the hedging performance of the our strategy to dynamic delta hedging with daily rebalancing under different controlled model environments, we find that under a one-factor diffusion environment the delta hedge strategy works well and generates comparable performance to our hedge strategy. However, when the underlying dynamics experience large unexpected jumps and/or stochastic volatility, the delta hedge strategy deteriorates markedly while our strategy remains stable and well performing. A historical hedging exercise on the SPX options further shows that as long as the target–hedge maturity distance is within 6 months, our hedge portfolio can perform significantly better than dynamic delta hedging.

Both dynamic hedging and our new proposal rely on Taylor expansion and term matching of the value functions between the target and the hedging instruments. Dynamic hedging expands along risk factors and forms hedges to match risk sensitivities whereas our approach expands along contract characteristics and matches contract features. The mathematical similarity dictates that the two approaches are similar in general applicability, not only to vanilla options but also to other derivative contracts. Nevertheless, the different expansion dimensions also lead to drastically different economic implications on hedging choice and performance stability. Expanding along risk factors for the dynamic hedging approach dictates that the hedger must know the true risk sources to generate the correct risk match, that the size of the hedging errors depends crucially on the size of the random shocks from the risk sources. Accordingly, the hedging error can be made extremely small if one can rebalance the hedge continuously, if all actual risks ex post are as assumed ex ante, and if they all move diffusively so that the higher-order expansion terms on random shocks are kept small. The hedge deteriorates or even breaks down with discrete rebalancing, when some unknown and hence unhedged risk sources show up, or when the risk moves by a randomly large amount. By contrast, expanding along contract characteristics under our approach dictates that instead of focusing on risk dynamics, the hedger must understand the
contract structures well and choose hedging instruments close in characteristics to the target contract to minimize the high-order expansion errors on characteristic differences. When contracts are well matched in characteristics, they stay matched over time, regardless of the underlying risk dynamics. Therefore, our approach pays more attention to contract features than risk dynamics, and the resulting hedge tends to be more stable over time and more robust to shocks and surprises in the underlying risk dynamics.

Our hedging theory is new and different from the existing literature. As such, it opens many interesting areas for future research. For example, more theoretical and numerical research can be devoted to the hedging error behaviors for different strike-maturity placements. Future research can also be directed at hedging strategies for an option portfolio that involves a large number of strikes and maturities. In this case, one can expand all options at some common reference strike and expiry and match expansion terms. To reduce risk further, one can also divide the options into several strike-maturity buckets, and match expansion terms within each bucket to reduce the expansion errors within each bucket. One can also combine the principle of hedging with nearby contracts with traditional risk sensitivity matching practices, thus adding robustness whenever possible to a flexible practice.

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References


