Hedging Barriers

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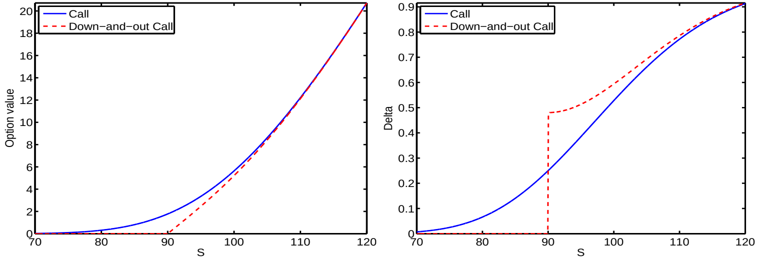
Hedging barriers: Overview

- Barrier options are actively traded in the OTC currency market.
- Hedging barriers faces two major challenges:
  - Compared to delta hedging of vanilla options, delta hedging of barriers is subject to larger errors *in practice*.
    —The errors are larger when the delta varies more over time.
  - Reliable barrier option quotes are hardly available, making performance comparison across different models/hedging strategies difficult.
    - The traditional approach: Simulation based on an assumed environment.

Comparing vanilla call to down-out call

Option values and delta (under Black-Scholes)

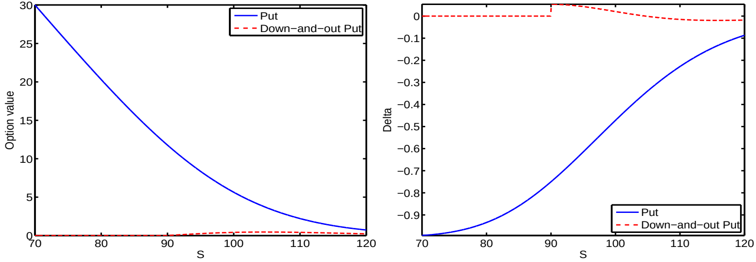


- Down-and-out: The option has zero value once the lower barrier is crossed.
- Delta drops to zero once the barrier is crossed.

Model parameters: $r = q = 0$, $\sigma = 20\%$, $K = 100$, $t = 6/12$, L=90.

Comparing vanilla put to down-out put

Option values and delta (under Black-Scholes)



- The terminal payoff (conditional on survival) increases with lowering spot, but the down-out feature works against it.

Model parameters: $r = q = 0, \sigma = 20\%, K = 100, t = 6/12$, L=90.

Hedging barriers: Objective

- I propose an approach, with which we can systematically examine the barrier hedging effectiveness of several strategies empirically
  - based on historical paths on the FX and vanilla FX option prices,
  - but *without resorting to barrier option quotes*.
- Two objectives
  - What is the best hedging strategy really?
    - How many dimensions to hedge.
    - Which models to use.
  - How many dimensions of risk do we need to model?

Tracking the P&L of a hedging strategy

- Consider N hedging strategies or models.
- At each date t, sell the target instrument for an *unknown* price P_t,
- Put on hedging positions on the underlying currency and/or vanilla currency options (straddles, risk reversals, butterflies).
- Let $\{H_t^n\}_{n=1}^N$ denotes the hedging cost of the N strategies, observable based on the market quotes on currency prices and vanilla option prices.
- Track the aggregate hedging cost to maturity, $\{H_{t+\tau}^n\}_{n=1}^N$.
- The terminal cost of selling the barrier, $P_{t+\tau}$, is observable based on the underlying currency path, even though the barrier price is not before expiry.
- Terminal P&L of hedging a barrier sold at time t with strategy n is: $PL_t^n = P_t e^{r\tau} - P_{t+\tau} - H_{t+\tau}^n$
- Compare the time-series variation of PL_t^n for different strategies.
- Issue: *How to calculate PL^n without observing the barrier selling price P_t?*

Hedging performance comparison

without observing the target instrument price

We consider two solutions, each with different underlying assumptions:

- Compute the P&L excluding the barrier sales revenue: Replace $PL_t^n = P_t e^{r\tau} - P_{t+\tau} - H_{t+\tau}^n$ with $\widetilde{PL}_t^n = -P_{t+\tau} - H_{t+\tau}^n$.
  - If P_t is constant over time, $Var(PL_t^n) = Var(\widetilde{PL}_t^n)$.
  - More generally, for hedging strategy m, n, $Var(PL_t^m) > Var(PL_t^n)$ if $Var(\widetilde{PL}_t^m) > Var(\widetilde{PL}_t^n)$ and the covariance terms do not dominate: $Cov(PL_t^m, P_t e^{r\tau}) - Cov(PL_t^n, P_t e^{r\tau}) < Var(PL_t^m) - Var(PL_t^n)$.
  - Choose a barrier contract whose value does not vary much over time.
- Compute the P&L using the model fair value as the selling price.
  - If there exists a model that generates the perfect hedging strategy, then the P&L computed based on the fair value of the model should always be zero.
  - To the point that the P&L is not zero and varies over time, it reflects both the pricing error and the hedging ineffectiveness.
- Good news: Both solutions generate the same ranking in terms of hedging performance.

Hedging exercise design

- Target:
  - One-touch paid at expiry with a lower barrier (OTPE).
  - Maturity: 30 calendar days.
  - Barrier (L) set to 25 Balck-Scholes put delta.
    - Considering barrier in terms of delta instead of percentage (98%) spot generates more stable barrier values.
  - Spot normalized to \$100. Notional set at \$100.
- Data and sample period:
  - Currencies: dollar-yen, dollar-pound, both with dollar as domestic.
  - Sample period: 1996/1/24–2004/1/28, 2927 days.
  - Instruments: currencies, delta-neutral straddle, 10- and 25-delta risk reversal and butterfly spread at 1 week, and 1,2,3,6,9, 12 months.

Hedging exercise design

- Procedure:
  - Sell a one touch at each date (2897 days) and monitor hedging (rebalancing if needed) over the next 30 days.
  - Set the selling price to (i) zero and (ii) the model fair value.
  - Store the two types of terminal P&L from the hedging exercise.
  - Report/compare the summary statistics on the P&L over the 2897 exercises for each currency pair across different hedging strategies.
  - Also report the summary statistics of no hedging.

Hedging strategies

- *Dynamic Strategies* (Rebalancing based on partial derivatives):
  - Hedging dimensions (sources of risks hedged):
    - Delta (spot sensitivity) using the underlying currency.
    - Vega (volatility/variance sensitivity) using delta-neutral straddle.
    - Vanna (spot/volatility cross sensitivity) using risk reversal.
    - Volga using butterfly spread.
  - Hedging models (used to compute the sensitivities):
    - Black-Scholes (Garman-Kohlhagen).
    - Heston stochastic volatility (daily parameter calibration).
    - Carr-Wu stochastic skew model (daily parameter calibration).
    - Dupire local volatility model
  - Rebalancing frequencies:
    - *Daily.*
    - Weekly.
    - Once (vanna-volga approach?).
- *Static strategy* (No rebalancing until barrier crossing or expiry)

The Black-Scholes model

- Black-Scholes model: $dS/S = (r - q)dt + \sigma dW$
  - Definitions of greeks:

$$Delta \equiv \frac{\partial P}{\partial S_t}, Vega \equiv \frac{\partial P}{\partial \sigma}, Vanna \equiv \frac{\partial^2 P}{\partial S \partial \sigma}, Volga \equiv \frac{\partial^2 P}{\partial \sigma \partial \sigma},$$

  - σ input in calculating the greeks is the $IV(L, \tau)$, obtained via linear interpolation across strikes.
  - When L is outside the available delta range, using the IV at the boundary (no extrapolation).
  - At maturities shorter than one month (as time goes by), use one month quotes on implied volatility and interest rates.
  - Delta, vega are analytical. Vanna and volga are numerical.

Pricing and hedging under the Black-Scholes model

- Vanilla and barrier pricing:

$$C_t(K,T) = S_t e^{-q(T-t)} N(d_1) - K e^{-r(T-t)} N(d_2),$$
$$OTPE_t(L,T) = e^{-r(T-t)} \left(N(-d_2) + \left(\frac{S_t}{L}\right)^{2p} N(-d_3) \right),$$

with
$p \equiv \frac{1}{2} - \frac{r-q}{\sigma^2}$, $d_{1,2} = \frac{\ln(S_t/K)+(r-q\pm\sigma^2/2)(T-t)}{\sigma\sqrt{T-t}}$, $d_3 = \frac{\ln(S_t/L)-(r-q-\sigma^2/2)(T-t)}{\sigma\sqrt{T-t}}$.

- Hedging ratios on vanillas and barriers:

$$Delta(C_t(K,T)) = e^{-q(T-t)} N(d_1), \quad Vega(C_t(K,T)) = S_t e^{-q(T-t)} \sqrt{T-t}\, n(d_1),$$
$$Vanna(C_t(K,T)) = -e^{-q(T-t)} n(d_1) d_2/\sigma.$$
$$Delta(OTPE_t(L,T)) = -e^{-r(T-t)} \left(n(d_2) d_2'(S_t) - \frac{2p}{L}\left(\frac{S_t}{L}\right)^{2p-1} N(-d_3) + \left(\frac{S_t}{L}\right)^{2p} n(d_3) d_3'(S_t) \right),$$
$$Vega(OTPE_t(L,T)) = e^{-r(T-t)} \left(n(d_2) d_4 + \left(\frac{S_t}{L}\right)^{2p} n(d_3) d_5 + \left(\frac{S_t}{L}\right)^{2p} N(-d_3) \ln\left(\frac{S}{L}\right) \frac{4(r-q)}{\sigma^3} \right),$$

with $n(x) \equiv e^{-x^2/}\sqrt{2\pi}$ denoting a normal density function and $d_2'(S) = d_3'(S) = 1/(S\sigma\sqrt{T-t})$, and

$$d_4 = \frac{\ln(S_t/L) + (r-q+\sigma^2/2)(T-t)}{\sigma^2\sqrt{T-t}}, \quad d_5 = \frac{\ln(S_t/L) - (r-q+\sigma^2/2)(T-t)}{\sigma^2\sqrt{T-t}}.$$

The Heston stochastic volatility model

- Heston model:
  $dS/S = (r-q)dt + \sqrt{v_t}dW, \quad dv_t = \kappa(\theta - v_t)dt + \sigma_v\sqrt{v_t}dZ, \rho dt = \mathbb{E}[dWdZ]$

  - Definitions of greeks:

$$Delta \equiv \frac{\partial P}{\partial S_t}, Vega \equiv \frac{\partial P}{\partial v_t}, Vanna \equiv \frac{\partial^2 P}{\partial S \partial v_t},$$

  - Model parameters and v_t are obtained from daily calibration to vanilla options (1m to 12m). κ is fixed to one.
  - Use finite-difference method to determine values and greeks for the barrier. Use FFT method to determine greeks for vanilla options.
  - Also consider adjusted delta:

$$\text{Adj. Delta} \quad \equiv \quad \frac{dOTPE_t(S_t)}{dS_t} = \frac{\partial OTPE_t(S_t)}{\partial S_t} + \frac{\partial OTPE_t(S_t)}{\partial v_t}\frac{\sigma_v \rho}{S_t}.$$

Pricing and hedging under Heston

- Under the Heston model, we can derive the conditional generalized Fourier transform of the log currency return $s_{t,\tau} = \ln S_{t+\tau}/S_t$ in closed form:

$$\phi_s(u) \equiv \mathbb{E}_t\left[e^{iu \ln S_{t+\tau}/S_t}\right] = e^{iu(r-q)\tau - a(\tau) - b(\tau)v_t}, \quad u \in \mathcal{D} \subseteq \mathbb{C}, \qquad (1)$$

  where

$$\begin{aligned} a(\tau) &= \tfrac{\kappa}{\sigma_v^2}\left[2\ln\left(1 - \tfrac{\eta-\widetilde{\kappa}}{2\eta}\left(1 - e^{-\eta\tau}\right)\right) + (\eta - \widetilde{\kappa})\tau\right], \\ b(\tau) &= \tfrac{2\psi\left(1-e^{-\eta\tau}\right)}{2\eta-(\eta-\widetilde{\kappa})(1-e^{-\eta\tau})}, \end{aligned} \qquad (2)$$

  and

$$\eta = \sqrt{(\widetilde{\kappa})^2 + 2\sigma_v^2\psi}, \quad \widetilde{\kappa} = \kappa - iu\rho\sigma\sigma_v, \quad \psi = \frac{1}{2}\sigma^2(iu + u^2).$$

- Given the Fourier transform, we can price European vanilla options numerically using fast Fourier inversion.

- To price barrier options, we use a finite-difference scheme.

The Carr-Wu stochastic skew model

- We implement a simplified (pure diffusion) version of the SSM:

$$\begin{aligned} dS/S &= (r-q)dt + \sqrt{v_t^R}dW^R + \sqrt{v_t^L}dW^L, \\ dv_t^j &= \kappa(\theta - v_t^j)dt + \sigma_v\sqrt{v_t^j}dZ^j, j = R, L, \rho^j dt = \mathbb{E}[dW^i dZ^j] \end{aligned}$$

The model generates not only stochastic volatility, but also stochastic skew (random risk reversal).

  - Definitions of greeks:

$$Delta \equiv \frac{\partial P}{\partial S_t}, Vega \equiv \frac{\partial P}{\partial (v_t^R + v_t^L)}, Vanna \equiv \frac{\partial^2 P}{\partial v_t^R \partial v_t^L},$$

  - Model parameters and v_t are obtained from daily calibration to vanilla options (1m to 12m). κ is fixed to one. $\rho^R = -\rho^L > 0$.
  - Use simulation to determine greeks for barrier. Use FFT method to determine greeks for vanilla options.

Pricing and hedging under the Carr-Wu stochastic skew model

- Under the stochastic skew model, the conditional generalized Fourier transform of the log currency return is also known in closed form:

$$\phi_s(u) = e^{iu(r-q)\tau - a^R(\tau) - b^R(\tau)v_t^R - a^L(\tau) - b^L(\tau)v_t^L}, \tag{3}$$

  where

$$\begin{aligned} a^j(\tau) &= \frac{\kappa}{\sigma_v^2}\left[2\ln\left(1 - \frac{\eta - \widetilde{\kappa}^j}{2\eta^j}\left(1 - e^{-\eta^j\tau}\right)\right) + (\eta^j - \widetilde{\kappa}^j)\tau\right], \\ b^j(\tau) &= \frac{2\psi\left(1 - e^{-\eta^j\tau}\right)}{2\eta^j - (\eta^j - \widetilde{\kappa}^j)\left(1 - e^{-\eta^j\tau}\right)}, \end{aligned} \tag{4}$$

  and

$$\eta^j = \sqrt{(\widetilde{\kappa}^j)^2 + 2\sigma_v^2\psi}, \quad \widetilde{\kappa}^j = \kappa - iu\rho^j\sigma\sigma_v, \quad \psi = \frac{1}{2}\sigma^2(iu + u^2).$$

- Given the generalized Fourier transform, we price European vanilla options numerically using fast Fourier inversion.

- We price the barrier option using Monte Carlo simulation.

The Dupire local volatility model

- Dupire (1994) local volatility model:

$$
\begin{aligned}
dS/S &= (r-q)dt + \sigma(S,t)dW, \\
\sigma^2(K,T) &= \frac{2\left(\frac{\partial C(K,T)}{\partial T} + (r-q)K\frac{\partial C(K,T)}{\partial K} + qC(K,T)\right)}{K^2\frac{\partial^2 C(K,T)}{\partial K^2}}.
\end{aligned}
$$

  - Definitions of greeks:

$$
Delta \equiv \frac{\partial P}{\partial S_t}, \; Vega \equiv \frac{\partial P}{\partial \sigma(S,t)}, \; Vanna \equiv \frac{\partial^2 P}{\partial S \partial \sigma(S,t)},
$$

  - Local volatility space implied from the vanilla implied volatility surface with one week to 12 month options.
  - Use finite difference method to determine greeks.

Implementation details on the local volatility model

- Based on the implied volatility quotes IV at each each strike and maturity, we first define the total implied variance as $w = IV^2 T$.
- We interpolate the total implied variance against maturity T and moneyness $k \equiv \ln K/F$ using cubic splines with matching second derivatives to obtain a smooth function $w(k, T)$.
- We then directly derive the local volatility in terms of this smooth total implied variance function:

$$\sigma^2(K, T) = \frac{w_T + (r-q)w_k}{1 - \frac{kw_k}{w} - \frac{w_k^2}{4}\left(\frac{1}{4} + \frac{1}{w} - \frac{k^2}{w^2}\right) + \frac{w_{kk}}{2}}, \tag{5}$$

where w_T, w_k, w_{kk} denote the partial derivative of the total variance against the option maturity and moneyness k, respectively.

The static hedging strategy: The idea

- Under the Black-Scholes model, the one-touch can be replicated by a European-style terminal payoff structure,

$$\text{Payoff}_T = 1(S_T < L)\left(1 + \left(\frac{S_T}{L}\right)^{2p}\right), \quad p = \frac{1}{2} - \frac{r_d - r_f}{\sigma^2}. \qquad (6)$$

- The *static* hedging strategy:
  - At time t, sell one barrier $OTPE_t(L, T)$ and put on a static hedging position with vanilla options with the terminal payoff equal to (6).
  - If the barrier is never hit before expiry, both the one-touch and the hedging portfolio generate zero payoff.
  - If the barrier is crossed at time $T_1 \in (t, T)$, rebalance the static hedging position to match the one dollar liability at expiry.
    - Sell a European option with the terminal payoff $1(S_T < L)\left(\frac{S_T}{L}\right)^{2p}$.
    - Buy a European option that pays $1(S_T > L)$.

    The operation is self-financing under the Black-Scholes model.
  - The net result of this operation is to generate a portfolio that pays one dollar regardless of the terminal stock price.

The static hedging strategy: Simplification

- Simplification: Set $p = \frac{1}{2}$ which is exact when $r = q$.
- The replicating portfolio becomes

$$\text{Payoff}_T = 1(S_T < L)\left(1 + \left(\frac{S_T}{L}\right)^{2p}\right) = 2(S_T < L) - \frac{(L - S_T)^+}{L}$$

  Long 2 binary puts, short $1/L$ put.

- The payoff is zero if the barrier is never hit.
- When the barrier is hit, we do the following:
  - Sell one binary put and buy $1/L$ put: $\left.\frac{\partial P_{T_1}(K,T)}{\partial K}\right|_{K=L} - \frac{P_{T_1}(L,T)}{L}$,
  - Buy a binary call: $-\left.\frac{\partial C_{T_1}(K,T)}{\partial K}\right|_{K=L}$.
  - The operation is self-financing under the model assumption.
  - The terminal payoff of this rebalanced portfolio would be one dollar, just like the one touch.
  - The strategy works even in the presence of stochastic volatility (as long as it is independent of currency return).

The static hedging strategy: Implementation

- We assume that we can readily trade the binary puts and binary calls.
- We explore two methods for valuation:
  - Take derivatives of the Black-Scholes formula against strike, $BP(L) = -\left.\frac{\partial BSP(K,IV)}{\partial K}\right|_{K=L}$, regarding implied volatility as a constant.
  - Adjust for the implied-volatility smile effect: $BP(L) = -\left.\frac{\partial BSP(K,IV(K))}{\partial K}\right|_{K=L} - \left.\frac{\partial BSP(K,IV(K))}{\partial IV}\frac{\partial IV(K)}{\partial K}\right|_{K=L}$.
- To obtain the derivative, we apply cubic spline on the implied volatility as a function of log strike, and then evaluate the derivative at the barrier.
- One touch valuation: $OTPE_t(L,T) = 2BP(L) - \frac{P_t(L,T)}{L}$.
- The payoff is zero if the barrier is never hit.
- When the barrier is hit, we sell one binary put, buy $1/L$ put, and buy a binary call to generate $1 payoff.
- The rebalancing is self-financing in theory, but can generate a cost/benefit when model assumptions are violated.
- We carry the actual rebalancing cost to maturity as terminal P&L.

Dynamic hedging on Black Scholes

Currency	JPYUSD				GBPUSD			
Strategy	Mean	Std	Skew	Kurt	Mean	Std	Skew	Kurt
	A. Terminal PL excluding sales from one touch							
No Hedge	-49.46	50.01	-0.02	1.00	-32.21	46.73	-0.76	1.58
Delta	-43.19	23.36	0.83	8.40	-36.49	21.13	1.30	7.65
Vega	-43.37	17.99	1.83	13.27	-38.86	17.91	2.06	11.24
Vanna	-45.32	18.32	1.65	12.47	-39.86	17.13	2.03	11.10
	B. Terminal PL including one-touch model value as revenue							
No Hedge	5.64	49.62	-0.02	1.00	17.71	46.61	-0.76	1.58
Delta	11.91	23.45	0.84	8.46	13.42	21.38	1.28	7.57
Vega	11.74	18.14	1.85	13.25	11.06	18.13	2.03	11.05
Vanna	9.79	18.43	1.66	12.59	10.05	17.35	2.01	10.91
	C. Model value with implied volatility at the barrier							
Value	54.91	1.86	0.11	2.43	49.73	1.09	0.06	2.19

Dynamic hedging on Heston

Strategy	Mean	Std	Skew	Kurt	Mean	Std	Skew	Kurt
	A. Terminal PL excluding sales from one touch							
No Hedge	-49.46	50.01	-0.02	1.00	-32.21	46.73	-0.76	1.58
Delta	-44.24	20.88	0.79	7.86	-36.39	19.81	1.20	6.34
Vega	-44.45	17.50	1.40	10.14	-38.24	16.58	1.85	8.82
Vanna	-45.38	18.01	1.32	10.08	-39.07	16.33	1.85	8.56
Adj. Delta	-44.24	20.88	0.79	7.86	-36.39	19.81	1.20	6.34
	B. Terminal PL including one-touch model value as revenue							
No Hedge	6.14	49.46	-0.02	1.02	17.32	46.54	-0.76	1.58
Delta	11.36	21.00	0.78	8.32	13.14	19.96	1.20	6.31
Vega	11.15	17.75	1.42	10.50	11.28	16.82	1.84	8.64
Vanna	10.22	18.27	1.32	10.47	10.46	16.57	1.83	8.41
Adj. Delta	11.36	21.00	0.78	8.32	13.14	19.96	1.20	6.31
	C. Model value with implied volatility at the barrier							
Value	55.40	3.71	0.55	2.77	49.35	1.66	0.21	4.09

Dynamic hedging on local volatility model

Strategy	Mean	Std	Skew	Kurt	Mean	Std	Skew	Kurt
	A. Terminal PL excluding sales from one touch							
No Hedge	-49.50	50.01	-0.02	1.00	-32.38	46.80	-0.75	1.57
Delta	-43.96	22.01	1.36	9.64	-36.38	20.86	1.35	7.87
	B. Terminal PL including one-touch model value as revenue							
No Hedge	4.21	49.62	-0.02	1.03	14.37	46.50	-0.75	1.57
Delta	9.75	22.05	1.42	10.02	10.37	20.92	1.36	7.89
	C. Model value with implied volatility at the barrier							
Value	48.62	9.75	-1.25	6.40	45.26	7.15	-1.34	8.06

Dynamic hedging on stochastic skew model

We still need to work on the numerical issues in computing the barrier hedging ratios.

Static hedging

Currency	JPYUSD				GBPUSD			
Strategy	Mean	Std	Skew	Kurt	Mean	Std	Skew	Kurt
	A. Terminal PL excluding sales from one touch							
Non-Adjusted	-50.86	10.68	6.66	62.00	-46.55	10.80	5.07	39.81
Smile-Adjusted	-45.83	12.86	4.30	30.60	-41.20	11.98	4.08	27.24
	B. Terminal PL including one-touch model value as revenue							
Non-Adjusted	0.55	10.69	6.63	61.74	4.53	10.80	5.07	39.87
Smile-Adjusted	2.62	13.05	4.13	28.40	4.92	12.17	4.10	26.54
	C. Model value with implied volatility at the barrier							
Non-Adjusted	51.23	0.40	1.75	9.28	50.90	0.21	-0.01	3.21
Smile-Adjusted	48.28	3.44	0.33	2.57	45.95	2.36	-0.31	3.06

Concluding remarks

- Among dynamic hedging strategies, delta hedging significantly reduces the P&L variation.
- Adding vega hedging on top of delta hedging also helps.
- But hedging additional risk dimension no longer generate visible reduction in risk.
- For each dynamic strategy considered, computing the greeks using Heston generates better performance than computing the greeks uing BS.
- The simple static strategy works better than all dynamic strategies.
- Remaining issue: *For pricing, which one is the most accurate?*