Where Do Heavy Tails Come From?
Disentangling the Multi-dimensional Variations in S&P 500 Index Options

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Joint work with Peter Carr

Conference on Latest Developments in Heavy-Tailed Distributions
March 27, 2010
Evidence: The distributions of financial security returns are heavy tailed.

- **What types of data-generating mechanism lead to these heavy tails?**
  - Understanding the mechanism becomes important when we look at
  - Vanilla options across different maturities and dates
    - Return aggregation across maturity — **CLT?**
    - Distribution variation across time — **IID?**
  - Related vanilla products such as variance swaps, VIX options...
    - How big should return variance be? — **Should we buy VS?**
    - How much does return variance vary?
  - Exotic (and not so exotic) options such as barrier options ...

- Statistically, there are at three 3 ways the tails can grow heavy.
  - Return innovations jump, e.g., $\alpha$-stable Lévy motion (or its dampened version).
  - Stochastic volatility innovation is correlated with return innovation, e.g., Heston.
  - Return volatility is a function of the price level, e.g., Dupire.

- **We want to understand which process generates what behavior.**
We also want understand the economic rationale behind the statistical process.

1 **The leverage effect**: With business risk fixed, an increase in financial leverage *level* leads to an increase in equity volatility *level*.
   - A financial leverage increase can come *passively* from stock price decline while the debt level is fixed — Black (76)'s classic leverage story.
   - It can also come *actively* from active leverage management.

2 **The volatility feedback effect** on asset valuation:
   - A positive *shock* to business risk increases the discounting of future cash flows, and reduces the asset value, regardless of the level of financial leverage.

3 **The self-exciting behavior** of market disruptions:
   - A downside jump in the stock (index) price leads to an upside spike in the chances of having more of the same.
The model

- Decompose the forward value of the equity index $F_t$ into a product of the asset value $A_t$ and the equity-to-asset ratio (EAR) $X_t$,

$$F_t = A_t X_t.$$ 

⇐ This is just a tautology,

- Separately model the (risk-neutral) dynamics of $X_t$ and asset value $A_t$.
  
  - Model $X_t$ as a CEV process: $dX_t / X_t = \delta X_t^{-p} dW_t, \quad p > 0$.
  
  - Leverage effect: A decline in $X$ reduces equity value, increases leverage, and raises equity volatility.
  
  - Model the asset value $A_t$ as an exponential martingale,

$$\frac{dA_t}{A_t} = \sqrt{v_t} dZ_t + \int_{-\infty}^{0} (e^x - 1) (\mu^+ (dx, dt) - \pi_J+(x)dx \nu_t^J dt) + \int_{0}^{\infty} (e^x - 1) (\mu^- (dx, dt) - \pi_J-(x)dx \nu_t^J dt),$$

$$d\nu_t^Z = \kappa_Z (\theta_Z - \nu_t^Z) dt + \sigma_Z \sqrt{v_t} dZ_t^\gamma, \quad \mathbb{E} [dZ_t^\gamma dZ_t] = \rho \ dt,$$

$$d\nu_t^J = \kappa_J (\theta_J - \nu_t^J) dt - \sigma_J \int_{-\infty}^{0} x (\mu^- (dx, dt) - \pi_J-(x)dx \nu_t^J dt).$$

  - Volatility feedback — $\rho < 0$.
  
  - Self-exciting crashes — $\sigma_J > 0$. Negative jumps in asset return are associated with positive jumps in the jump arrival rate $\nu_t^J$.  

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Why separately model asset value and leverage?

- Financial leverages may not increase when asset prices are down.
- Firms actively manage their leverages (Adrian & Shin (2008)):
  - Commercial banks try to keep their leverage constant.
  - Investment banks take larger leverages during booming periods and de-lever during recessions.
- The traditional leverage story (on the negative relation between stock returns and volatilities) work better for households (and to a lesser degree manufacturing companies) with passive capital structure managements.
- We use the model for pricing SPX options, but the same logic can also be used, more naturally, on single name options.
  - Do firms following different financial leverage decision rules show different option pricing behaviors on their stocks?
How to model large market disruptions?

- Models log price with $\alpha$-stable Lévy motion $\rightarrow$ $\alpha$-stable distribution.
  - Does not work if the right tail is $\alpha$-stable in $\mathbb{Q}$ (Merton).

- How does market price stable risk? (JH McCulloch): It cannot be stable both ways (need to be tilted/convoluted).

- Central limit theorem works in time series, but not in options (Wu, 2006) — Evidence later.
  - Dampened $\alpha$-stable power law (DPL):
    \[
    \pi_{J^+}(x) = \lambda e^{-x/v_{J^+}} x^{-\alpha-1}, \quad \pi_{J^-}(x) = \lambda e^{-|x|/v_{J^-}} |x|^{-\alpha-1}.
    \]
    Finite variance and higher moments.

- Risk aversion leads to negative skew in options (less dampening on down jumps) even if time-series return is symmetric.

- When risk aversion is at its maximum, dampening on down jumps disappears, downside return becomes pure $\alpha$-stable, return variance becomes infinite for options pricing, even though it is finite for time series return.

- DLP is our choice here for modeling market disruptions in both return and volatility.
The stock price dynamics with price level dependence

- The stock price dynamics:

\[
\frac{dF_t}{F_t} = \delta \left( \frac{F_t}{A_t} \right)^{-p} dW_t + \sqrt{v_t^Z} dZ_t \\
+ \int_{R_0} (e^x - 1) \left( \mu(dx, dt) - \pi_J(x) dx v_t^J dt \right).
\]

- The stock return variance depends on the price level.
- With \( p > 0 \), the return variance increases with declining price level.
- Scaling \( F_t \) by \( A_t \) (both in dollars) makes the return variance a unitless quantity,
- and renders the dynamics \textit{scale free} and stable in the presence of splits or trends.

- In addition to the level dependence, \((A_t, v_t^Z, v_t^J)\) add separate variations to the stock return variance.
The stock price dynamics without level dependence

- An alternative representation:

\[
\frac{dF_t}{F_t} = \delta X_t^{-p} dW_t + \sqrt{\nu_t^Z} dZ_t \\
+ \int_{R_0} (e^x - 1) \left( \mu(dx, dt) - \pi_J(x) dx v_t^J dt \right).
\]

- Return variance is driven by three state variables \((X_t, \nu_t^Z, \nu_t^J)\), with no additional level dependence.

- Let \(\nu_t^X = \delta^2 X_t^{-2p}\), we obtain a three-factor stochastic volatility model:

\[
\frac{dF_t}{F_t} = \sqrt{\nu_t^X} dW_t + \sqrt{\nu_t^Z} dZ_t \\
+ \int_{R_0} (e^x - 1) \left( \mu(dx, dt) - \pi_J(x) dx v_t^J dt \right).
\]

where

\[
d\nu_t^X = \kappa_X (\nu_t^X)^2 dt - \sigma_X (\nu_t^X)^{3/2} dW_t, \Leftarrow \text{a 3/2-process.}
\]

with \(\kappa_X = p(2p + 1)\) and \(\sigma_X = 2p\). Henceforth, normalize \(\delta = 1\).

- The model can be represented either as a local vol model with level dependence or a pure scale-free stochastic volatility model without level dependence — unifying the two strands of literature.
Market prices of risks and statistical dynamics

- The specifications so far are on the risk-neutral dynamics:
  - The forward price, forward asset value, and leverage ratio $X_t$ are all assumed to be martingales under $\mathbb{Q}$.
  - Their deviations from the $\mathbb{P}$ dynamics reflect the market prices of risks.
  - Managers make financial leverage decisions according to the current levels of all three types of risks:

$$dX_t = X_t^{1-p} \left( a_X - \kappa_{XX} X_t - \kappa_{XZ} v_t^Z - \kappa_{XJ} v_t^J \right) dt + X_t^{1-p} dW_t^P.$$  

Market price of $W_t$ risk is $\gamma_t^X = a_X - \kappa_{XX} X_t - \kappa_{XZ} v_t^Z - \kappa_{XJ} v_t^J$.

- $\kappa_{XX}$: Mean reversion, leverage level targeting.
- $\kappa_{XZ}$: Response to diffusion business risk.
- $\kappa_{XJ}$: Response to jump business risk.

- Constant market prices ($\gamma^v, \gamma^J$) for diffusion variance risk ($Z_t$) and jump risk ($J_t$).
Consider the forward value of a European call option:

\[
c(F_t, K, T) = \mathbb{E}_t \left[ (F_T - K)^+ \right]
= \mathbb{E}_t \left[ \mathbb{E}_t \left[ (X_{T}A_{T} - K)^+ \mid X_T = \mathcal{X} \right] \mid X_t \right]
= \mathbb{E}_t \left[ \mathcal{X} \cdot C(A_t, K/\mathcal{X}, T) \mid X_t \right]
\]

where \( C(A_t, K, T) \equiv \mathbb{E} [(A_T - K)^+] \) is the forward call value on asset.

Option valuation follows a two-step numerical procedure:

- Derive the Fourier transform of the asset return. Apply fast Fourier inversion (FFT) to compute the call value on asset \( C \). — Order \( N \ln(N) \) computation.

- Integrate the call value \( \mathcal{X}C \) over the known density of \( X_T = \mathcal{X} \) conditional on \( X_t \):

\[
f(X_T = \mathcal{X}, X_t) = \frac{\mathcal{X}^{2p-3} X_t^{1/2}}{p(\tau - t)} \exp \left( -\frac{X_t^{2p} + \mathcal{X}^{2p}}{2p^2(\tau - t)} \right) I_v \left( \frac{X_t^{p} \mathcal{X}^{p}}{p^2(\tau - t)} \right), v = \frac{1}{2p}
\]
— Quadrature method.
Data analysis


- 40 time series on a grid of
  - 5 relative strikes: 80, 90, 100, 110, 120% of spot.
  - 8 fixed time to maturities: 1m, 3m, 6m, 1y, 2y, 3y, 4y, 5y.

- Listed market focuses on short-term options (within 3 years). OTC market is very active on long-dated options.

- At one maturity, an implied volatility smile/skew can be generated by many different mechanisms — all you can learn is a heavy-tailed distribution.

- To distinguish the different roles played by the different mechanisms, we need to look at how these smiles/skews evolve across a wide range of maturities and over different time periods.
Implied volatilities show a negatively sloped skew along strike.

⇔ *Return distribution has a down-side heavy tail.*

The skew slope becomes flatter as maturity increases due to scaling: 80% strike at 5-yr maturity is not nearly as out of money as 80% strike at 1-month maturity.

When measured against a standardized moneyness measure $d = \ln(K/100)/(IV \sqrt{\tau})$, the skew defined as,

$$SK_{t,T} = \frac{IV_{t,T}(80\%) - IV_{t,T}(120\%)}{|d_{t,T}(80\%) - d_{t,T}(120\%)|},$$

does NOT flatten as maturity increases.

⇔ *Central limit theorem does NOT kick in up to 10 years.*
Time series behavior: *IID?*

The S&P 500 Index

ATM implied volatility, %

Implied volatility term structure, %

Implied volatility skew, %
3 PCs explain 96.6% of variation: 85.1%, 8.2%, 3.3%.

- The 1st PC (blue solid line) — the average volatility level variation.
- The 2nd PC (green dashed) — the variation in the term structure.
- The 3rd PC (red dash-dotted) — the variation along strike.

The ranking of the 2nd & 3rd PCs can switch for listed options data as the listed market has more quotes along strikes than maturities.
1. Stochastic volatility $\leftarrow (X_t, \nu^Z_t, \nu^J_t)$
2. Stochastic term structure $\leftarrow$ Different term structure responses to shocks from $(X_t, \nu^Z_t, \nu^J_t)$.
3. Stochastic skew $\leftarrow$ variations of $\nu^Z_t$ versus $(X_t, \nu^J_t)$. 

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An negative shock to index return is instantaneously associated with

- a positive shock to the volatility level \((-0.8114)\)
- a steepening of the skew \((-0.707)\)
- a flattening of the term structure \([TS=5y-1mATMV]\) \((0.7643)\)

Over-reaction — Reversion in ATMV, SK, and TS one week later.

Long-run prediction — High vol/skew predicts high return in 2 months.
Self-exciting behavior: Implied volatility and skew respond more to large downside index jumps than upside index jumps.
Model estimation, with dynamic consistency

- The model has 10 parameters ($p, \kappa_Z, \theta_Z, \sigma_Z, \rho, \kappa_J, \theta_J, \sigma_J, v_J^+, v_J^-$) and three hidden state variables ($X_t, v_Z^t, v_J^t$) to price 40 options each date.

- We fix the model parameters over time and use the 3 state variables to capture the variation of the $5 \times 8$ implied volatility surface.

- We cast the model into a state-space form:
  
  - Let $V_t \equiv [X_t, v_Z^T, v_J^T]$ be the state. State propagation equation: $V_t = f(V_{t-1}; \Theta) + \sqrt{Q_{t-1}} \varepsilon_t$.
  - 6 additional parameters ($a, \kappa_{XX}, \kappa_{XZ}, \kappa_{XJ}, \gamma^v, \gamma^J$).

  - Let the 40 option series be the observation. Measurement equation: $y_t = h(V_t; \Theta) + \sqrt{R} e_t$, $(40 \times 1)$
    - $y$: OTM option prices scaled by the BS vega of the option.
    - Assume that the pricing errors on the scaled option series are iid.

  - Estimate 17 parameters over 23,320 options (11 years, 583 weeks, 40 options each day), using (quasi) maximum likelihood method joint with unscented Kalman filter.
Pricing performance

<table>
<thead>
<tr>
<th>$\mathcal{K}/\tau$</th>
<th>1</th>
<th>3</th>
<th>6</th>
<th>12</th>
<th>24</th>
<th>36</th>
<th>48</th>
<th>60</th>
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<tbody>
<tr>
<td>Root mean squared pricing error in volatility points: Average=0.83</td>
<td></td>
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<tr>
<td>80</td>
<td>2.216</td>
<td>1.103</td>
<td>1.050</td>
<td>0.836</td>
<td>0.607</td>
<td>0.550</td>
<td>0.770</td>
<td>1.064</td>
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<tr>
<td>90</td>
<td>1.225</td>
<td>0.727</td>
<td>0.701</td>
<td>0.641</td>
<td>0.445</td>
<td>0.279</td>
<td>0.445</td>
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<tr>
<td>100</td>
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<td>0.418</td>
<td>0.286</td>
<td>0.377</td>
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<td>0.720</td>
<td>0.717</td>
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<td>0.465</td>
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<td>120</td>
<td>4.014</td>
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<td>1.057</td>
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<td>0.714</td>
<td>0.561</td>
<td>0.572</td>
<td>0.735</td>
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<td>$R^2$: Average=95.6%</td>
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<tr>
<td>80</td>
<td>0.897</td>
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<td>0.961</td>
<td>0.970</td>
<td>0.983</td>
<td>0.987</td>
<td>0.971</td>
<td>0.936</td>
</tr>
<tr>
<td>90</td>
<td>0.965</td>
<td>0.985</td>
<td>0.982</td>
<td>0.982</td>
<td>0.990</td>
<td>0.996</td>
<td>0.989</td>
<td>0.965</td>
</tr>
<tr>
<td>100</td>
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<td>0.994</td>
<td>0.991</td>
<td>0.986</td>
<td>0.991</td>
<td>0.996</td>
<td>0.992</td>
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<td>110</td>
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<td>0.979</td>
<td>0.977</td>
<td>0.974</td>
<td>0.984</td>
<td>0.989</td>
<td>0.989</td>
<td>0.978</td>
</tr>
<tr>
<td>120</td>
<td>0.293</td>
<td>0.938</td>
<td>0.939</td>
<td>0.945</td>
<td>0.971</td>
<td>0.981</td>
<td>0.981</td>
<td>0.973</td>
</tr>
</tbody>
</table>

The errors are on average within the bid-ask spreads.
Average variance contributions

- Average return variance contributions from 3 different sources:

<table>
<thead>
<tr>
<th>Source</th>
<th>Notation</th>
<th>Variance</th>
<th>Vol</th>
</tr>
</thead>
<tbody>
<tr>
<td>Financial Leverage</td>
<td>$\mathbb{E}^P [X_t^{-2p}]$</td>
<td>0.0119</td>
<td>10.91%</td>
</tr>
<tr>
<td>Diffusion Business risk</td>
<td>$\mathbb{E}^P [\nu_t^2]$</td>
<td>0.0231</td>
<td>15.19%</td>
</tr>
<tr>
<td>Jump Business risk</td>
<td>$\mathbb{E}^P [\left( \nu_{j+}^2 + \nu_{j-}^2 \right) \nu_t^j]$</td>
<td>0.0116</td>
<td>10.79%</td>
</tr>
</tbody>
</table>
### Average skew contributions from 4 different sources:

<table>
<thead>
<tr>
<th>Source</th>
<th>Symmetric null</th>
<th>Estimates</th>
</tr>
</thead>
<tbody>
<tr>
<td>down/up Jump</td>
<td>$v_{J^-} = v_{J^+}$</td>
<td>$v_{J^-} = 0.1926 \gg v_{J^+} \approx 0$</td>
</tr>
<tr>
<td>Leverage effect</td>
<td>$p = 0$</td>
<td>$p = 2.8427$</td>
</tr>
<tr>
<td>Volatility feedback</td>
<td>$\rho = 0$</td>
<td>$\rho = -0.8354$</td>
</tr>
<tr>
<td>Self-excitement</td>
<td>$\sigma_J = 0$</td>
<td>$\sigma_J = 5.6355$</td>
</tr>
</tbody>
</table>
1. **Leverage** $v_X^t$: Mean-repelling (drift=$p(2p + 1)(v_X^t)^2 dt$)  
   \[ \Rightarrow \text{Responses to shocks become larger at longer maturities.} \]

2. **Diffusion business risk** $v_Z^t$: Strong mean reversion ($\kappa_Z = 3.0114$)  
   \[ \Rightarrow \text{Responses decline quickly as option maturity increases.} \]

3. **Jump business risk** $v_J^t$: Slow mean reversion ($\kappa_J = 0.0009$)  
   \[ \Rightarrow \text{Responses do not decline.} \]

\[ \Rightarrow \text{The impacts are } v_Z^t \text{ (volatility feedback) are mainly an short term options. }\]
\[ X_t \text{ (leverage) and } v_J^t \text{ (self-exciting jump) extend to long-term options.} \]
Implied volatility skew (heavy left tail) variations

1. **Leverage** $X_t$: High leverage increases volatility and long-term skew, but reduces mid-term skew.

2. **Diffusion business risk** $\nu^Z_t$: High diffusion risk increases short-term volatility, and intermediate skew (through volatility feedback).

3. **Jump business risk** $\nu^J_t$: High jump risk increases both volatility and skew at both short and long maturities.
   - Short-term skew increase is due to increase in negative jumps.
   - Long-term skew increase is due to self-excitement (volatility feedback on drugs).
The capital structure decision

\[
dX_t = X_t^{1-p} \left( a_X - \kappa_{XX} X_t - \kappa_{XZ} v^Z_t - \kappa_{XJ} v^J_t \right) dt + X_t^{1-p} dW^P_t
\]

<table>
<thead>
<tr>
<th>Θ</th>
<th>Estimates</th>
<th>Std Error</th>
</tr>
</thead>
<tbody>
<tr>
<td>(a_X)</td>
<td>0.0003</td>
<td>0.0000</td>
</tr>
<tr>
<td>(\kappa_{XX})</td>
<td>0.0001</td>
<td>0.0000</td>
</tr>
<tr>
<td>(\kappa_{XZ})</td>
<td>17.5360</td>
<td>0.3087</td>
</tr>
<tr>
<td>(\kappa_{XJ})</td>
<td>-0.0774</td>
<td>0.0000</td>
</tr>
</tbody>
</table>

- \(\kappa_{XX} = 0.0001\): Capital structure is very persistent.
- \(\kappa_{XZ} = 17.536\): High diffusion business risk reduces \(X_t\) and hence increases the financial leverage.
- \(\kappa_{XJ} = -0.0774\): High jump business risk increases \(X_t\) and hence reduces the financial leverage.

⇒ The key concern of financial leverage is default/crash (sustainability), not daily fluctuations — Levering up increases your fluctuation, but also increases your return, ... if only you can survive. Potentially better stories on different types of single-name companies ...
The risk contribution from financial leverage ($v_t^X$) reached historical highs before the burst of the Nasdaq bubble.

The diffusion variance risk ($v_t^Z$) peaked during the 2003 recession.

The jump risk ($v_t^J$) spiked during the LTCM crisis.
Wherever you look for heavy tails, you find them. Now what?

It is important to understand the different channels through which heavy tails are generated and how each channel affects the pricing and hedging of derivatives differently.

Option prices across different strikes, maturities, and time also provide a lot more information about the different channels than does the underlying return.

It is helpful to model the variation of the financial leverage and the business risk separately
- to bridge the gaps in the literature,
- to disentangle the different mechanisms of interactions, and
- to generate good pricing performance on equity options over both short and long option maturities.

The approach has potentials in analyzing single name stock options.
- Link the different capital structure management styles to the different behaviors of the implied volatility surface.
"Heavy" Hangman

Self-exciting crash (Suddenly increased 50 pounds, hang myself!)

Leverage (Junk Food)

Volatility Feed back (Depressed and hence start to each more)

Jump (Genetics)