

# Leverage Effect, Volatility Feedback, and Self-Exciting Market Disruptions

Disentangling the Multi-dimensional Variations in S&P 500 Index Options

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# Co-movements between stock index and index volatilities

Equity index and volatilities show negative co-movements.

- Mechanisms that can generate such co-movements:
  - Add negative instantaneous correlation between innovations in index return and return variance, e.g., Heston (1993).
    - *Scale-free dynamics*: Changing the units/scale of the index does not change the dynamics.
  - Model the volatility as a function of the index level.
    - The local volatility model of Dupire (1994):
$$dS_t = (r - q)S_t dt + \sigma(S_t, t)dW.$$
    - The constant elasticity of variance:  $\sigma(S_t, t) = \delta S_t^{1-p}$ ,  $p > 0$ .
    - Scaling the price level alters the dynamics.
- Evidence:
  - Derman (1999): Data show different regimes, under which the implied volatility and the equity index show different dependence structures.

# Our take: multiple channels of interactions

Index and index volatility interacts through at least three distinct channels:

- ① *The leverage effect*: With business risk fixed, an increase in financial leverage level leads to an increase in equity volatility level.
  - A financial leverage increase can come from stock price decline while the debt level is fixed — Black (76)'s classic leverage story.
  - It can also come from active leverage management.
- ② *The volatility feedback effect* on asset valuation:
  - A positive shock to business risk increases the discounting of future cash flows, and reduces the asset value, regardless of the level of financial leverage.
- ③ *The self-exciting behavior* of market disruptions:
  - A downside jump in the index leads to an upside spike in the chances of having more of the same.

# An overview of what we do

- We propose a model for the stock index dynamics that captures all three channels of interactions,
  - by modeling the dynamics of the asset value and the dynamics of the financial leverage separately.
- We propose a tractable way of pricing options under the specified dynamics.
- We estimate the model on 12 years of S&P 500 index options.
  - and compare its performance with that of the state-of-the-art reduced-form benchmark.
- We explore the implications of the different interaction channels on the variation of the implied volatility surface.

# The model

- Decompose the forward value of the equity index  $F_t$  into a product of the asset value  $A_t$  and the equity-to-asset ratio  $X_t$ ,

$$F_t = A_t X_t. \quad \Leftarrow \text{This is just a tautology,}$$

- Separately* model the dynamics of leverage  $X_t$  and asset value  $A_t$ .

- Model  $X_t$  as a CEV process:  $dX_t/X_t = \delta X_t^{-p} dW_t$ ,  $p > 0$ .
  - Leverage effect**: A decline in  $X$  reduces equity value, increases leverage, and raises equity volatility.
- Model the asset value  $A_t$  as an exponential martingale,

$$\begin{aligned} \frac{dA_t}{A_t} &= \sqrt{v_t^Z} dZ_t + \int_0^\infty (e^x - 1) (\mu^+(dx, dt) - \pi_{J^+}(x) dx v_t^J dt) \\ &\quad + \int_{-\infty}^0 (e^x - 1) (\mu^-(dx, dt) - \pi_{J^-}(x) dx v_t^J dt), \\ dv_t^Z &= \kappa_Z (\theta_Z - v_t^Z) dt + \sigma_Z \sqrt{v_t^Z} dZ_t^v, \quad \mathbb{E}[dZ_t^v dZ_t^v] = \rho dt, \\ dv_t^J &= \kappa_J (\theta_J - v_t^J) dt - \sigma_J \int_{-\infty}^0 x (\mu^-(dx, dt) - \pi_{J^-}(x) dx v_t^J dt). \end{aligned}$$

- Volatility feedback** —  $\rho < 0$ .
- Self-exciting crashes** — Negative jumps in asset return are associated with positive jumps in the arrival rate of jumps  $v_t^J$ .
- VG jumps**:  $\pi_{J^+}(x) = e^{-x/v_{J^+}} x^{-1}$ ,  $\pi_{J^-}(x) = e^{-|x|/v_{J^-}} |x|^{-1}$ .

# Why separately model asset value and leverage?

- Financial leverages may not increase when stock prices are down.
- Firms actively manage their leverages (Adrian & Shin (2008)):
  - Commercial banks try to keep their leverage constant.
  - Investment banks take larger leverages during booming periods and de-lever during recessions.
- The traditional leverage story (on the negative relation between stock returns and volatilities) work better for households (and to a lesser degree manufacturing companies) with passive capital structure managements.
- The *state of the art in option pricing*: Directly model the equity index with two stochastic volatility factors: Bates (2000), Jones (2003), Huang & Wu (2004), Carr & Wu (2007):
  - By setting  $X = 1$ , and  $F_t = A_t$ , we generate a reduced-form *benchmark* that supercedes all the above:

$$\frac{dF_t}{F_t} = \sqrt{v_t^Z} dZ_t + \int_0^\infty (e^x - 1) (\mu^+(dx, dt) - \pi_{J^+}(x) dx v_t^J dt) + \int_{-\infty}^0 (e^x - 1) (\mu^-(dx, dt) - \pi_{J^-}(x) dx v_t^J dt).$$

# The equity index dynamics with index level dependence

- The equity index dynamics:

$$dF_t/F_t = \delta \left( \frac{F_t}{A_t} \right)^{-p} dW_t + \sqrt{v_t^Z} dZ_t + \int_{\mathbb{R}_0} (e^x - 1) (\mu(dx, dt) - \pi_J(x) dx v_t^J dt).$$

- The index return variance depends on the index level.
- With  $p > 0$ , the return variance increases with declining index level.
- Scaling  $F_t$  by  $A_t$  (both in dollars) makes the return variance a unitless quantity,
- and renders the dynamics *scale free* and stable in the presence of splits or trends.
  - We cannot expect the long-run index level to be stable, but we can expect the level of financial leverage to be stationary.
- In addition to the level dependence,  $(A_t, v_t^Z, v_t^J)$  add separate variations to the index return variance.

# The equity index dynamics *without* index level dependence

- An alternative representation:

$$dF_t/F_t = \delta X_t^{-p} dW_t + \sqrt{v_t^Z} dZ_t + \int_{\mathbb{R}_0} (e^x - 1) (\mu(dx, dt) - \pi_J(x) dx v_t^J dt).$$

- Index return variance is driven by three state variables  $(X_t, v_t^Z, v_t^J)$ , with *no additional index level dependence*.
- Let  $v_t^X = \delta^2 X_t^{-2p}$ , we obtain a three-factor stochastic volatility model:

$$dF_t/F_t = \sqrt{v_t^X} dW_t + \sqrt{v_t^Z} dZ_t + \int_{\mathbb{R}_0} (e^x - 1) (\mu(dx, dt) - \pi_J(x) dx v_t^J dt).$$

where

$$dv_t^X = \kappa_X (v_t^X)^2 dt - \sigma_X (v_t^X)^{3/2} dW_t, \Leftarrow \text{a } 3/2\text{-process.}$$

with  $\kappa_X = p(2p + 1)$  and  $\sigma_X = 2p$ . Henceforth, normalize  $\delta = 1$ .

- The model can be represented either as a local vol model with index level dependence or a pure stochastic volatility model with no index level dependence — unifying the two strands of literature.

- Consider the forward value of a European call option:

$$\begin{aligned}c(F_t, K, T) &= \mathbb{E}_t \left[ (F_T - K)^+ \right] \\ &= \mathbb{E}_t \left[ \mathbb{E}_t \left[ (\mathcal{X}A_T - K)^+ \mid (X_T = \mathcal{X}) \mid X_t \right] \right] \\ &= \mathbb{E}_t \left[ \mathcal{X} \cdot \mathcal{C}(A_t, K/\mathcal{X}, T) \mid X_t \right]\end{aligned}$$

where  $\mathcal{C}(A_t, K, T) \equiv \mathbb{E}[(A_T - K)^+]$  is the forward call value on asset.

- Option valuation follows a two-step procedure:
  - Derive the Fourier transform of the asset return. Apply fast Fourier inversion (FFT) to compute the call value on asset  $\mathcal{C}$ . — Order  $N \ln(N)$  computation.
  - Integrate the call value  $\mathcal{X}\mathcal{C}$  over the known density of  $X_T = \mathcal{X}$  conditional on  $X_t$ :

$$f(X_T = \mathcal{X}, X_t) = \frac{\mathcal{X}^{2p - \frac{3}{2}} X_t^{\frac{1}{2}}}{\rho(T-t)} \exp\left(-\frac{X_t^{2p} + \mathcal{X}^{2p}}{2\rho^2(T-t)}\right) I_\nu\left(\frac{X_t^p \mathcal{X}^p}{\rho^2(T-t)}\right), \nu = \frac{1}{2p}$$

— Quadrature method.

# Market prices of risks and statistical dynamics

- The specifications so far are on the risk-neutral dynamics:
  - The forward index level, forward asset value, and leverage ratio  $X_t$  are all martingales under  $\mathbb{Q}$ .
- Their deviations from the  $\mathbb{P}$  dynamics reflect the market prices of risks.
- We postulate that managerial decisions on financial leverage depend on risk levels:

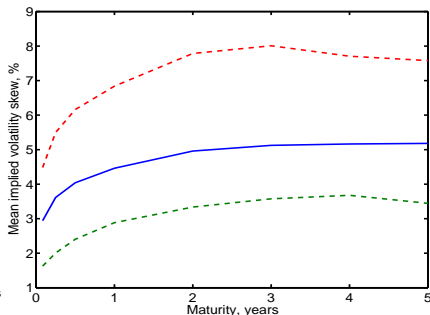
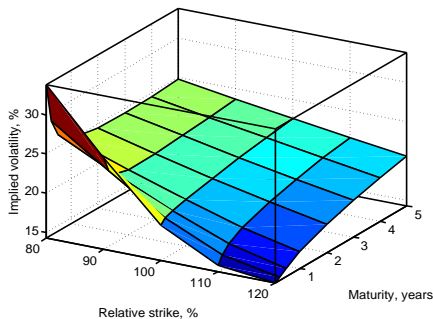
$$dX_t = X_t^{1-p} (a_X - \kappa_{XX}X_t - \kappa_{XZ}v_t^Z - \kappa_{XJ}v_t^J) dt + X_t^{1-p} dW_t^{\mathbb{P}}.$$

Market price of  $W_t$  risk is  $\gamma_t^X = a_X - \kappa_{XX}X_t - \kappa_{XZ}v_t^Z - \kappa_{XJ}v_t^J$ .

- $\kappa_{XX}$ : Mean reversion, leverage level targeting.
  - $\kappa_{XZ}$ : How leverage responds to diffusion business risk.
  - $\kappa_{XJ}$ : How leverage responds to jump business risk.
- Constant market prices on diffusion variance risk ( $\gamma^V$ ) and jump risk ( $\gamma^J$ ).

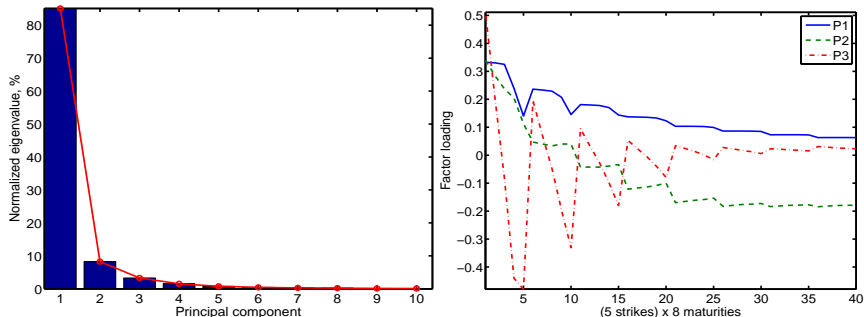
- OTC implied volatility quotes on SPX options, January 1996 to March 2008, 583 weeks.
- 40 time series on a grid of
  - 5 relative strikes: 80, 90, 100, 110, 120% of spot.
  - 8 fixed time to maturities: 1m, 3m, 6m, 1y, 2y, 3y, 4y, 5y.
- Listed market focuses on short-term options (within 3 years).  
OTC market is very active on long-dated options.
  - At one maturity, an implied volatility smile/skew can be generated by many different mechanisms: jumps, leverage, volatility feedback, self-exciting crashes...
  - To distinguish the different roles played by the different mechanisms, we need to look at how these smiles/skews evolve over a wide range of maturities.

# The mean implied volatility surface and skew



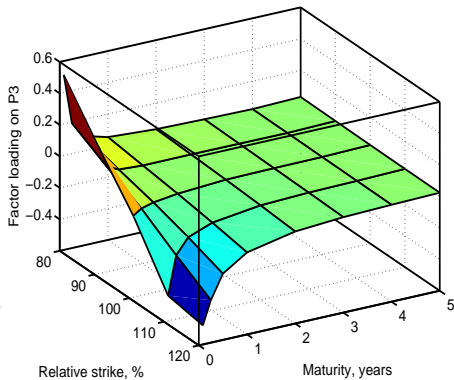
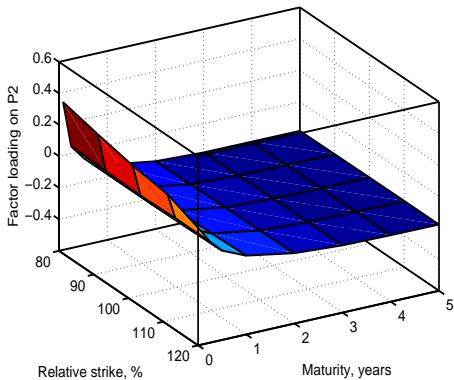
- Implied volatilities show a negatively sloped skew along strike.
- The skew slope becomes flatter as maturity increases due to scaling: 80% strike at 5-yr maturity is not nearly as out of the money as 80% strike at 1-month maturity.
- When measured against a standardized moneyness measure  $d = \ln(\mathcal{K}/100)/(IV\sqrt{T})$ , the skew defined as, 
$$SK_{t,T} = \frac{IV_{t,T}(80\%) - IV_{t,T}(120\%)}{|d_{t,T}(80\%) - d_{t,T}(120\%)|}$$
, does not flatten as maturity increases.

# Principal component analysis



- 3 PCs explain 96.6% of variation: 85.1%, 8.2%, 3.3%.
  - The 1st PC (blue solid line) — the average volatility level variation.
  - The 2nd PC (green dashed) — the variation in the term structure.
  - The 3rd PC (red dash-dotted) — the variation along strike.
- The ranking of the 2nd & 3rd PCs can switch for listed options data as the listed market has more quotes along strikes than maturities.

# Principal component loadings on the implied vol surface



# Model estimation, with dynamic consistency

- The model has 10 parameters ( $\rho, \kappa_Z, \theta_Z, \sigma_Z, \rho, \kappa_J, \theta_J, \sigma_J, v_{J+}, v_{J-}$ ) and three hidden state variables ( $X_t, v_t^Z, v_t^J$ ) to price 40 options each date.
- The PCA suggests that the majority of the implied volatility surface variation can be captured by three properly placed components.
- We fix the model parameters over time and use the 3 state variables to capture the variation of the  $5 \times 8$  implied volatility surface.
- We cast the model into a state-space form:
  - Let  $V_t \equiv [X_t, v_t^Z, v_t^J]$  be the state. State propagation equation:  
$$V_t = f(V_{t-1}; \Theta) + \sqrt{Q_{t-1}} \varepsilon_t.$$
  
— 6 additional parameters ( $a, \kappa_{XX}, \kappa_{XZ}, \kappa_{XJ}, \gamma^V, \gamma^J$ ).
  - Let the 40 option series be the observation. Measurement equation:  $y_t = h(V_t; \Theta) + \sqrt{R} e_t$ , ( $40 \times 1$ )
    - $y$ : OTM option prices scaled by the BS vega of the option.
    - Assume that the pricing errors on the scaled option series are iid.
  - Estimate 17 parameters over 23,320 options, using maximum likelihood method joint with unscented Kalman filter.

# Pricing performance: Root mean squared error

$\mathcal{K}/\tau$	1	3	6	12	24	36	48	60
Reduced-form benchmark								
80	3.904	2.155	1.515	1.301	1.131	0.902	0.908	1.181
90	2.252	1.112	0.852	0.810	0.708	0.591	0.711	1.056
100	1.290	0.601	0.641	0.678	0.528	0.425	0.592	0.966
110	2.486	0.982	1.067	1.025	0.711	0.488	0.554	0.901
120	3.836	1.488	1.299	1.327	1.014	0.663	0.583	0.856
Full model								
80	2.216	1.103	1.050	0.836	0.607	0.550	0.770	1.064
90	1.225	0.727	0.701	0.641	0.445	0.279	0.445	0.758
100	1.111	0.409	0.474	0.555	0.418	0.286	0.377	0.657
110	1.499	0.720	0.720	0.717	0.551	0.436	0.465	0.674
120	4.014	1.081	1.057	0.984	0.714	0.561	0.572	0.735

Average rmse (in vol points) is 1.187 for benchmark and 0.83 for full model.

# Pricing performance: Explained variation

$\mathcal{K}/\tau$	1	3	6	12	24	36	48	60
Reduced-form benchmark								
80	0.672	0.904	0.919	0.932	0.948	0.960	0.954	0.931
90	0.869	0.962	0.972	0.975	0.981	0.982	0.973	0.951
100	0.957	0.988	0.983	0.981	0.990	0.991	0.981	0.962
110	0.795	0.966	0.949	0.949	0.978	0.988	0.983	0.968
120	0.289	0.878	0.907	0.904	0.947	0.976	0.980	0.969
Average	0.717	0.940	0.946	0.948	0.969	0.979	0.974	0.956
Full model								
80	0.897	0.972	0.961	0.970	0.983	0.987	0.971	0.936
90	0.965	0.985	0.982	0.982	0.990	0.996	0.989	0.965
100	0.968	0.994	0.991	0.986	0.991	0.996	0.992	0.977
110	0.930	0.979	0.977	0.974	0.984	0.989	0.989	0.978
120	0.293	0.938	0.939	0.945	0.971	0.981	0.981	0.973
Average	0.810	0.974	0.970	0.971	0.984	0.990	0.984	0.966

Likelihood: 38,265 for benchmark and 46,651 for full model.

# Relative variance and skew contributions

$\Theta$	Estimates	Std Error
$p$	2.8427	0.0074
$\rho$	-0.8354	0.0016
$\sigma_J$	5.6355	0.0272
$v_{J+}$	0.0000	0.0000
$v_{J-}$	0.1926	0.0002

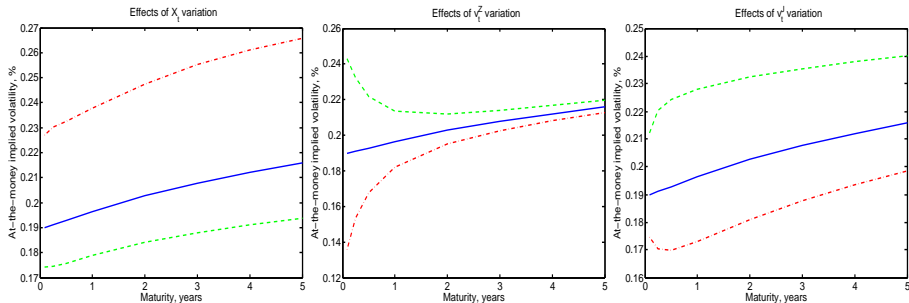
- Average instantaneous return variance (volatility) contributions from  $(X_t, v_t^Z, v_t^J)$ :  
 $\mathbb{E}^{\mathbb{P}}[X_t^{-2p}] = 0.0119(10.91\%),$   
 $\mathbb{E}^{\mathbb{P}}[v_t^Z] = 0.0231(15.19\%),$   
 $\mathbb{E}^{\mathbb{P}}[(v_{J+}^2 + v_{J-}^2)v_t^J] = 0.0116(10.79\%).$
- All three types of interactions are strong: Leverage effect ( $p = 2.8427$ ), volatility feedback ( $\rho = -0.8354$ ), and self-excitement ( $\sigma_J = 5.6355$ ).
- Much larger downside jumps than upside jumps ( $v_{J-} \geq v_{J+}$ ).

# Different term structure effects

$\Theta$	Estimates	Std Error
$\rho$	2.8427	0.0074
$\kappa_Z$	3.0114	0.0127
$\kappa_J$	0.0009	0.0000

- Different risk-neutral dynamics  $\Rightarrow$  different term structure responses:
  - $v_t^X$ : Mean-repelling (drift= $\rho(2\rho + 1)(v_t^X)^2 dt$ )  
 $\Rightarrow$  Responses to shocks become larger at longer maturities.
  - $v_t^Z$ : Strong mean reversion ( $\kappa_Z = 3.0114$ )  
 $\Rightarrow$  Responses decline quickly as option maturity increases.
  - $v_t^J$ : Slow mean reversion ( $\kappa_J = 0.0009$ )  
 $\Rightarrow$  Responses do not decline.

# The term structure of at-the-money implied volatility

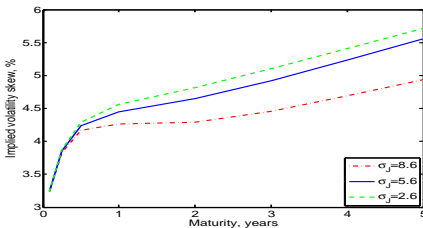
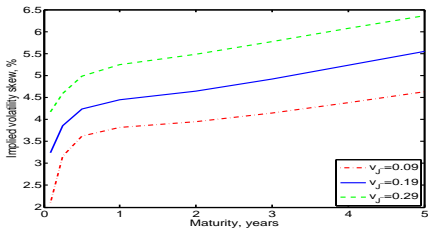
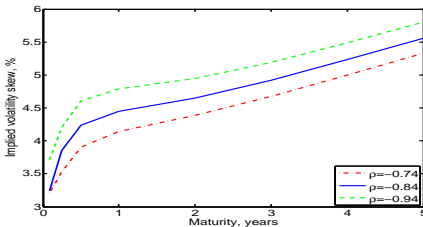
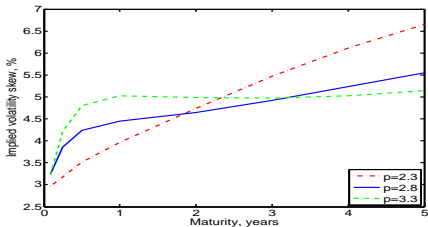


- Solid lines: Evaluated at the sample average of  $(X_t, v_t^Z, v_t^J)$ .
- Dashed lines: Evaluated at 90th-percentile for one state variable.
- Dashed lines: Evaluated at 10th-percentile for one state variable.

# Multiple sources of implied volatility skew

Our model embeds 4 mechanisms in generating the implied volatility skew:

- (1) leverage effect ( $\rho > 0$ ),
- (2) volatility feedback ( $\rho < 0$ ),
- (3) crash risk ( $v_{J-} \gg v_{J+}$ ),
- (4) self-excitement of crash events ( $\sigma_J$ ).



$$SK_{t,T} = (IV_{t,T}(80\%) - IV_{t,T}(120\%)) / (|d_{t,T}(80\%) - d_{t,T}(120\%)|)$$

# The capital structure decision

$$dX_t = X_t^{1-p} (a_X - \kappa_{XX}X_t - \kappa_{XZ}v_t^Z - \kappa_{XJ}v_t^J) dt + X_t^{1-p} dW_t^{\mathbb{P}}$$

$\Theta$	Estimates	Std Error
$a_X$	0.0003	0.0000
$\kappa_{XX}$	0.0001	0.0000
$\kappa_{XZ}$	17.5360	0.3087
$\kappa_{XJ}$	-0.0774	0.0000

- $\kappa_{XX} = 0.0001$ : Capital structure is very persistent.
- $\kappa_{XZ} = 17.536$ : High *diffusion* business risk reduces  $X_t$  and hence *increases* the financial leverage.
- $\kappa_{XJ} = -0.0774$ : High *jump* business risk increases  $X_t$  and hence *reduces* the financial leverage.

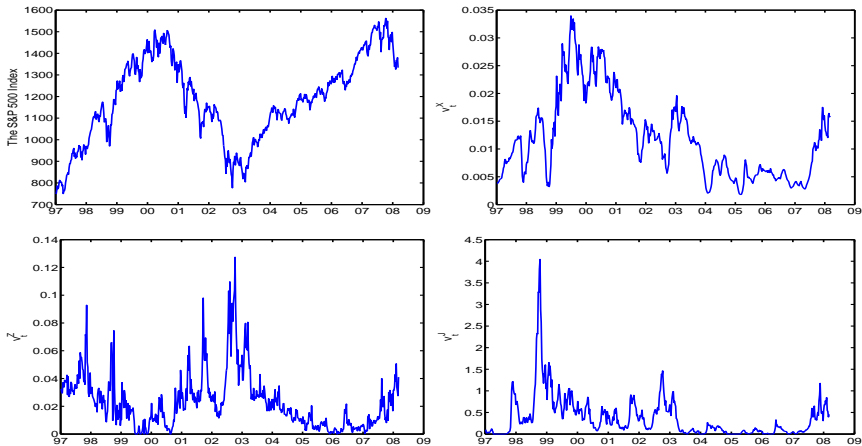
Potentially better stories on different types on individual companies ...

# Market prices of risks

$\Theta$	Estimates	Std Error
$\gamma^v$	-17.4507	0.3048
$\gamma^J$	0.4468	0.0023

- (i) Market price of diffusion variance risk ( $\gamma^v$ ) is highly negative  $\rightarrow$  Negative variance risk premium.
- Market price of return jump risk ( $\gamma^J$ ) is positive:
  - Risk-neutral return innovation distribution is more *negatively* skewed than the statistical distribution:  
$$v_{J+}^{\mathbb{P}} = v_{J+} / (1 - \gamma^J v_{J+}) > v_{J+},$$
$$v_{J-}^{\mathbb{P}} = v_{J-} / (1 + \gamma^J v_{J-}) < v_{J-}.$$
  - (ii) Instantaneous variance contribution from jumps is larger under the risk-neutral measure than under the statistical measure:  
$$(v_{J+}^2 + v_{J-}^2)v_t^J > ((v_{J+}^{\mathbb{P}})^2 + (v_{J-}^{\mathbb{P}})^2)v_t^J.$$
  - (iii) Negative risk premium on the jump arrival rate ( $v_t^J$ ):  
$$\sigma_J(v_{J-}^{\mathbb{P}} - v_{J-})v_t^J < 0.$$
- (i), (ii), & (iii)  $\Rightarrow$  Negative index return variance risk premium.

# Time series of the extracted states



- The risk contribution from financial leverage ( $v_t^X$ ) reached historical highs before the burst of the Nasdaq bubble.
- The diffusion variance risk ( $v_t^Z$ ) peaked during the 2003 recession.
- The jump risk ( $v_t^J$ ) spiked during the LTCM crisis.

# Concluding remarks

- Equity return and volatility interact through several distinct channels.
- It is helpful to separately model the variations of the financial leverage and the business risk
  - to bridge the gaps in the literature,
  - to disentangle the different mechanisms of interactions, and
  - to generate good pricing performance on equity options over both short and long option maturities.
- The approach has potentials in analyzing single name stocks options.
  - Link the different capital structure management styles to the different behaviors of the implied volatility surface.