Leverage Effect, Volatility Feedback, and Self-Exciting Market Disruptions
Disentangling the Multi-dimensional Variations in S&P 500 Index Options

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Co-movements between stock index and index volatilities

Equity index and volatilities show negative co-movements.

- Mechanisms that can generate such co-movements:
  - Add negative instantaneous correlation between innovations in index return and return variance, e.g., Heston (1993).
  - *Scale-free dynamics*: Changing the unitsSCALE of the index does not change the dynamics.
  - Model the volatility as a function of the index level.
    - The local volatility model of Dupire (1994):
      \[ dS_t = (r - q)S_t dt + \sigma(S_t, t)dW. \]
    - The constant elasticity of variance: \( \sigma(S_t, t) = \delta S_t^{1-p}, p > 0. \)
    - Scaling the price level alters the dynamics.
  - Evidence:
    - Derman (1999): Data show different regimes, under which the implied volatility and the equity index show different dependence structures.
Our take: multiple channels of interactions

Index volatility varies through at least three distinct channels:

1. The dependence is on the *level of financial leverage*:
   - Holding the business risk fixed, an increase in financial leverage level leads to an increase in equity volatility level.
     - A financial leverage increase can come from stock price decline while the debt level is fixed — Black (76)’s classic leverage story.
     - It can also come from active leverage management.

2. There is a separate “*volatility feedback*” effect on asset valuation, regardless of the level of financial leverage:
   - A positive shock to business risk increases the discounting of future cash flows, and reduces the asset value.

3. In addition, there are *self-exciting market disruptions*:
   - A downside jump in the index leads to an upside spike in the chances of having more of the same.
What we do

- We propose a model for the stock index dynamics that captures all three channels of interactions,
  - by modeling the dynamics of the asset value and the dynamics of the financial leverage separately.

- We propose a tractable way of pricing options under the specified dynamics.

- We estimate the model on 12 years of S&P 500 index options.
  - and compare its performance with that of the state-of-the-art reduced-form benchmark.

- We explore the implications of the different interaction channels on the variation of the implied volatility surface.
The model

- Decompose the forward value of the equity index $F_t$ into a product of the asset value $A_t$ and the equity-to-asset ratio $X_t$,

\[ F_t = A_t X_t. \]

This is just a tautology,

- But it allows us to separately model the dynamics of the $X_t$ and $A_t$:
  - Model $X_t$ as a CEV process: \( dX_t / X_t = \delta X_t^{-p} dW_t, \quad p > 0. \)
    - **Leverage effect**: A decline in $X$ reduces equity value, increases leverage, and raises equity volatility.
  - Model the asset value $A_t$ as an exponential martingale,
    \[
    \frac{dA_t}{A_t} = \sqrt{v^Z_t} dZ_t + \int_0^\infty (e^x - 1) \left( \mu^+(dx, dt) - \pi_{J+}(x) dx \nu^J_t dt \right) dt \\
    + \int_{-\infty}^0 (e^x - 1) \left( \mu^-(dx, dt) - \pi_{J-}(x) dx \nu^J_t dt \right) dt,
    \]
    \[
    dv^Z_t = \kappa_Z \left( \theta_Z - v^Z_t \right) dt + \sigma_Z \sqrt{v^Z_t} dZ^\nu_t, \quad \mathbb{E} [dZ^\nu_t dZ_t] = \rho dt,
    \]
    \[
    dv^J_t = \kappa_J \left( \theta_J - v^J_t \right) dt - \sigma_J \int_{-\infty}^0 x \left( \mu^-(dx, dt) - \pi_{J-}(x) dx \nu^J_t dt \right) dt.
    \]
  - **Volatility feedback** — $\rho < 0$.
  - **Self-exciting crashes** — Negative jumps in asset return are associated with positive jumps in the arrival rate of jumps $\nu^J_t$. 

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Why separately model asset value and leverage?

- Financial leverages may not increase when stock prices are down.
- Firms actively manage their leverages (Adrian & Shin (2008)):
  - Commercial banks try to keep their leverage constant.
  - Investment banks take larger leverages during booming periods and de-lever during recessions.
- The traditional leverage story (on the negative relation between stock returns and volatilities) work better for manufacturing companies with the level of debt relatively fixed.
  - By setting $X = 1$, and $F_t = A_t$, we generate a reduced-form benchmark that supercedes all the above:

$$\frac{dF_t}{F_t} = \sqrt{v_t^Z} dZ_t + \int_0^\infty (e^x - 1) (\mu^+(dx, dt) - \pi^+(x)dxv_t^J dt) + \int_{-\infty}^0 (e^x - 1) (\mu^-(dx, dt) - \pi^-(x)dxv_t^J dt).$$
A different notation for the asset value dynamics

- We can write the log asset return \( \ln A_T/A_t \) as a summation of two time-changed Lévy processes,

\[
\ln A_T/A_t = \left[ Z_{\mathcal{T}_T, T} - \frac{1}{2} \mathcal{T}_T Z_{T, T} \right] + \left[ J_{\mathcal{T}_J, T} - k_J(1) \mathcal{T}_J T, T \right],
\]

- Start with two types of movements as captured by the Lévy components \( Z_t \) (Brownian motion) and \( J_t \) (jumps).
- Apply separate time changes \( (\mathcal{T}_T Z_{T, T}, \mathcal{T}_J T, T) \) to the Lévy components.
- The time changes are defined via their respective activity rates,

\[
\mathcal{T}_T Z_{T, T} \equiv \int_t^T \nu_s^Z ds, \quad \mathcal{T}_J T, T \equiv \int_t^T \nu_s^J ds,
\]

- Brownian innovations in \( \nu_s^Z \) (variance rate) are negatively correlated with Brownian innovations in return \( dZ_t \).
- Jumps in \( \nu_s^J \) (arrival rate) are driven by downside jumps in return.
- Jump specification: Variance gamma with Lévy densities:

\[
\pi_{J^+}(x) = e^{-x/\nu_{J^+}} x^{-1}, \quad \pi_{J^-}(x) = e^{-|x|/\nu_{J^-}} |x|^{-1}.
\]
The equity index dynamics with index level dependence

- The equity index dynamics:

\[
\frac{dF_t}{F_t} = \delta \left( \frac{F_t}{A_t} \right)^{-p} dW_t + \sqrt{v_t} dZ_t + \int_{\mathbb{R}_0} (e^x - 1) \left( \mu(dx, dt) - \pi_J(x) dX_t \right) v_t J_t dt.
\]

- The index return variance depends on the index level.
- With \( p > 0 \), the return variance increases with declining index level.
- Scaling \( F_t \) by \( A_t \) (both in dollars) makes the return variance a unitless quantity,
- and renders the dynamics \textit{scale free} and stable in the presence of splits or trends.
  - We cannot expect the long-run index level to be stable, but we can expect the level of financial leverage to be stationary.
- In addition to the level dependence, \((A_t, v_t^Z, v_t^J)\) add separate variations to the index return variance.
The equity index dynamics without index level dependence

- An alternative representation:
  \[
  \frac{dF_t}{F_t} = \delta X_t^{-p} dW_t + \sqrt{v_t^Z} dZ_t \\
  + \int_{\mathbb{R}_0} (e^x - 1) \left( \mu(dx, dt) - \pi_J(x) dx v_t^J dt \right).
  \]

- Index return variance is driven by three state variables \((X_t, v_t^Z, v_t^J)\), with no additional index level dependence.

- Let \(v_t^X = \delta^2 X_t^{-2p}\), we obtain a three-factor stochastic volatility model:
  \[
  \frac{dF_t}{F_t} = \sqrt{v_t^X} dW_t + \sqrt{v_t^Z} dZ_t \\
  + \int_{\mathbb{R}_0} (e^x - 1) \left( \mu(dx, dt) - \pi_J(x) dx v_t^J dt \right).
  \]

  where
  \[
  dv_t^X = \kappa_X (v_t^X)^2 dt - \sigma_X (v_t^X)^{3/2} dW_t, \Leftarrow 3/2\text{-process}.
  \]

  with \(\kappa_X = p(2p + 1)\) and \(\sigma_X = 2p\). Henceforth, normalize \(\delta = 1\).

- The model can be represented either as a local vol model with index level dependence or a pure stochastic volatility model with no index level dependence — unifying the two strands of literature.
Consider the forward value of a European call option:

\[
c(F_t, K, T) = \mathbb{E}_t \left[ (F_T - K)^+ \right]
\[
= \mathbb{E}_t \left[ \mathbb{E}_t \left[ (X_A T - K)^+ \mid X_T = X \right] \mid X_t \right]
\[
= \mathbb{E}_t \left[ X \cdot C(A_t, K/X, T) \mid X_t \right]
\]

where \( C(A_t, K, T) \equiv \mathbb{E} \left[ (A_T - K)^+ \right] \) is the forward call value on asset.

Option valuation follows a two-step procedure:

- Derive the Fourier transform of the asset return. Apply fast Fourier inversion (FFT) to compute the call value on asset \( C \). — Order \( N \ln(N) \) computation.
- Integrate the call value \( X C \) over the known density of \( X_T = X \) conditional on \( X_t \):

\[
f(X_T = X, X_t) = \frac{X^{2p-3/2} X_t^{1/2}}{p(T-t)} \exp \left( -\frac{X_t^{2p} + X^{2p}}{2p^2(T-t)} \right) I_v \left( \frac{X_t^p X^{p}}{p^2(T-t)} \right).
\]

— Quadrature method.
The Fourier transform is exponential affine in \((v_t^Z, v_t^J)\):

\[
\phi(u) \equiv \mathbb{E}_t[e^{iu \ln A_T/A_t}] = \mathbb{E}_t \left[ e^{iu \left( Z_{T,t,T} - \frac{1}{2} T_{t,T}^Z \right) + iu \left( J_{T,t,T} - T_{t,T}^J k_J(1) \right) } \right] 
\]

\[
= \exp (-a_Z(\tau) - b_Z(\tau)v_t^Z - a_J(\tau) - b_J(\tau)v_t^J), \quad \tau = T - t,
\]

where the coefficients satisfy the following ODEs:

\[
b'_Z(\tau) = \psi_Z(u) - \kappa \sigma_Z^2 b_Z(\tau) - \frac{1}{2} \sigma_Z^2 b_Z(\tau)^2, \quad a'_Z(\tau) = b_Z(\tau) \kappa_Z \theta_Z,
\]

\[
b'_J(\tau) = \psi_J(u) - \kappa_J b_J(\tau) - k_{J^-}^M (b_J(\tau) \sigma_J), \quad a'_J(\tau) = b_J(\tau) \kappa_J \theta_J,
\]

starting at \(a_Z(0) = b_Z(0) = a_J(0) = b_J(0) = 0\), and

\[
\kappa_Z^M = \kappa_Z - iu \rho \sigma_Z, \quad k_{J^-}^M(s) = - \ln(1 + sv_{J^-}^M), \quad v_{J^-}^M = \frac{v_{J^-}}{1 + iuv_{J^-}}.
\]
FFT for option pricing

Let \( c(k) = C(A_t, K, T)/A_t \) with \( k = \ln K/A_t \).

Derive the Fourier transform of \( c(k) \) in terms of \( \phi(u) \) (asset return):

\[
\chi(u) \equiv \int_{-\infty}^{\infty} e^{iku} c(k) dk = \frac{\phi(u - i)}{(iu)(iu + 1)}, \quad u = u_r - i\nu, \nu > 0.
\]

The inversion:

\[
c(k) = \frac{e^{-\nu k}}{\pi} \int_0^{\infty} e^{-iu_r k} \chi(u_r - i\nu) du_r \approx \frac{e^{-\nu k}}{\pi} \sum_{m=0}^{N} \delta_m e^{-imu_k} \chi(u_m - i\nu) \Delta u.
\]

FFT: \( d_j = \frac{1}{N} \sum_{m=0}^{N-1} f_m e^{-jm^2\pi^2/N} \). If \( N \) is power of 2, \( N^2 \rightarrow (N/2) \log_2 N \).

Map the inversion to the FFT form by setting \( \eta = \Delta u \) and \( u_m = \eta m, k_j = -b + \lambda j \) with \( \lambda = 2\pi/(\eta N) \) and \( b = \lambda N/2 \).

\[
c(k_j) \approx \frac{1}{N} \sum_{m=0}^{N-1} f_m e^{jm^2\pi^2/N} , \quad f_m = \delta_m \frac{N}{\pi} e^{-\nu k_j + imu_m b} \chi(u_m - i\nu) \eta. \quad (1)
\]
Quadrature integration

- Approximate the integration with summation

\[
c(F_t, K, T) = \int_0^\infty f(X|X_t)XC(A_t, K/X, T)dX \approx \sum_{j=1}^{M} W_j X_jC(A_t, K/X_j, T),
\]

- Choose the points \( X_j \) and weights \( W_j \) based on Gauss-Hermite quadrature rule.

- Given the quadrature nodes and weights, \( \{x_i, w_j\}_{j=1}^{M} \), we set

\[
X_j = X_t e^{\sqrt{2V_X}x_j - \frac{1}{2}V_X}, \quad V_X = X_t^{-2p}(T-t),
\]

based on a log-normal approximation of the CEV dynamics.

- The summation weights are

\[
W_j = \frac{f(X_j|X_t)X'(x_j)}{e^{-x_j^2}} w_j = \frac{f(X_j|X_t)X_j\sqrt{2V_X}}{e^{-x_j^2}} w_j.
\]
The specifications so far are on the risk-neutral dynamics:

- The forward index level, forward asset value, and leverage ratio $X_t$ are all martingales under $\mathbb{Q}$.

- Their deviations from the $\mathbb{P}$ dynamics reflect the market prices of risks.

We postulate that managerial decisions on financial leverage depend on risk levels:

$$dX_t = X_t^{1-p} \left( a_X - \kappa_{XX} X_t - \kappa_{XZ} v_t^Z - \kappa_{XJ} v_t^J \right) dt + X_t^{1-p} dW_t^\mathbb{P}.$$ 

Market price of $W_t$ risk is $\gamma_t^X = a_X - \kappa_{XX} X_t - \kappa_{XZ} v_t^Z - \kappa_{XJ} v_t^J$.

- $\kappa_{XX}$: Mean reversion, leverage level targeting.
- $\kappa_{XZ}$: How leverage responds to diffusion business risk.
- $\kappa_{XJ}$: How leverage responds to jump business risk.

Constant market prices on diffusion variance risk ($\gamma^v$) and jump risk ($\gamma^J$).
Data analysis


- 40 time series on a grid of
  - 5 relative strikes: 80, 90, 100, 110, 120% of spot.
  - 8 fixed time to maturities: 1m, 3m, 6m, 1y, 2y, 3y, 4y, 5y.

- Listed market focuses on short-term options (within 3 years).
  OTC market is very active on long-dated options.

- At one maturity, an implied volatility smile/skew can be generated by many different mechanisms: jumps, leverage, volatility feedback, self-exciting crashes...

- To distinguish the different roles played by the different mechanisms, we need to look at how these smiles/skews evolve over a wide range of maturities.
Implied volatilities show a negatively sloped skew along strike.

The skew slope becomes flatter as maturity increases due to scaling: 80% strike at 5-yr maturity is not nearly as out of money as 80% strike at 1-month maturity.

When measured against a standardized moneyness measure
\[ d = \ln(\frac{K}{100})/\left(IV \sqrt{\tau} \right), \]
the skew defined as,
\[ SK_{t,T} = \frac{IV_{t,T}(80\%) - IV_{t,T}(120\%)}{|dt_{t,T}(80\%) - dt_{t,T}(120\%)|}, \]
does not flatten as maturity increases.
Downward sloping std term structure: Presence of a highly mean-reverting volatility factor.

Upward sloping auto term structure: Presence of multiple factors with different persistence.
3 PCs explain 96.6% of variation: 85.1%, 8.2%, 3.3%.

- The 1st PC (blue solid line) — the average volatility level variation.
- The 2nd PC (green dashed) — the variation in the term structure.
- The 3rd PC (red dash-dotted) — the variation along strike.

The ranking of the 2nd & 3rd PCs can switch for listed options data as the listed market has more quotes along strikes than maturities.
An negative shock to index return is instantaneously associated with
- a positive shock to the volatility level ($-0.8114$)
- a steepening of the skew ($-0.707$)
- a flattening of the term structure [TS=5y-1mATMV] ($0.7643$)

Over-reaction — Reversion in ATMV, SK, and TS one week later.

Long-run prediction — High vol/skew predicts high return in 2 months.
Self-exciting behavior: Implied volatility and skew respond more to large downside index jumps than upside index jumps.
Time series behavior

- The S&P 500 Index
- ATM implied volatility, %
- Implied volatility term structure, %
- Implied volatility skew, %

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The model has 10 parameters ($p, \kappa_Z, \theta_Z, \sigma_Z, \rho, \kappa_J, \theta_J, \sigma_J, v_{J+}, v_{J-}$) and three hidden state variables ($X_t, v^Z_t, v^J_t$) to price 40 options each date.

The PCA suggests that the majority of the implied volatility surface variation can be captured by three properly placed components.

We fix the model parameters over time and use the 3 state variables to capture the variation of the $5 \times 8$ implied volatility surface.

We cast the model into a state-space form:

- Let $V_t \equiv [X_t, v^Z_t, v^J_t]$ be the state. State propagation equation: $V_t = f(V_{t-1}; \Theta) + \sqrt{Q_{t-1}} \varepsilon_t$.
- 6 additional parameters ($a, \kappa_{XX}, \kappa_{XZ}, \kappa_{XJ}, \gamma^v, \gamma^J$) to control the statistical dynamics.

- Let the 40 option series be the observation. Measurement equation: $y_t = h(V_t; \Theta) + \sqrt{R}_t \varepsilon_t$, $(40 \times 1)$
  - $y$: OTM option prices scaled by the BS vega of the option.
  - Assume that the pricing errors on the scaled option series are iid.

Estimation 17 parameters over 23,320 options.
The Classic Kalman filter

- Under linear-Gaussian state-space setup,

\[
\begin{align*}
\text{State:} & \quad V_t = FV_{t-1} + \sqrt{Q} \epsilon_t, \\
\text{Measurement:} & \quad y_t = HV_t + \sqrt{R} e_t,
\end{align*}
\]

Kalman filter (KF) generates efficient forecasts and updates.

- The ex ante predictions are

\[
\begin{align*}
\bar{V}_t &= F\hat{V}_{t-1}, \\
\bar{y}_t &= H\bar{X}_t,
\end{align*}
\]

\[
\begin{align*}
\Sigma_{VV,t} &= F\Sigma_{VV,t-1}F^\top + Q, \\
\Sigma_{yy,t} &= H\Sigma_{VV,t}H^\top + R.
\end{align*}
\]

- The ex post filtering updates are (Bayes rule),

\[
\begin{align*}
K_t &= \Sigma_{VV,t}H^\top (\Sigma_{yy,t})^{-1} = \Sigma_{Vy,t} (\Sigma_{yy,t})^{-1}, \rightarrow \text{Kalman gain} \\
\hat{V}_t &= \bar{V}_t + K_t (y_t - \bar{y}_t), \\
\hat{\Sigma}_{VV,t} &= \Sigma_{VV,t} - K_t \Sigma_{yy,t} K_t^\top.
\end{align*}
\]

- We can build the log likelihood on the forecasting errors,

\[
l_t = -\frac{1}{2} \log |\Sigma_{yy,t}| - \frac{1}{2} \left( (y_t - \bar{y}_t)^\top (\Sigma_{yy,t})^{-1} (y_t - \bar{y}_t) \right).
\]
Approximating the distribution

Nonlinear measurement: \[ y_t = h(V_t) + \sqrt{Re_t} \]

- Extended KF: Linearly approximate the measurement equation: \[ h(V_t) \approx H_t V_t. \]

- Particle filter: Draw a large amount of random numbers and propagate these numbers using Bayes rule.

- Unscented filter: Use a few deterministically chosen “sigma” points to approximate the distribution.
  
  - More accurate than EKF, faster than UKF.
  - Let \( k \) be the number of states and \( \eta > 0 \) be a control parameter, we can generate a set of \( 2k + 1 \) sigma points \( \chi \) based on the mean \( V \) and covariance \( \Sigma_{VV} \) of the state:

  \[ \chi_0 = V, \quad \chi_i = V \pm \sqrt{(k + \eta)(\Sigma_{VV})_j}, \quad j = 1, \ldots, k; i = 1, \ldots, 2k, \]

  with weights \( w_i \) given by \( w_0 = \eta/(k + \eta), \quad w_i = 1/[2(k + \eta)]. \)

  - We can regard these sigma points as forming a discrete distribution with \( w_i \) as the corresponding probabilities.
The unscented Kalman filter

- At each time $t$, generate a set of $2k + 1$ sigma points $\chi_{t-1}$ based on time $(t - 1)$ updated state mean and covariance $\hat{V}_{t-1}$ & $\hat{\Sigma}_{VV,t-1}$.

- Given the sigma points, predict the time-$t$ state mean and covariance:
  \[
  \overline{\chi}_{t,i} = f(\chi_{t-1,i}; \Theta), \quad \overline{V}_t = \sum_{i=0}^{2k} w_i \overline{\chi}_{t,i},
  \]
  \[
  \overline{\Sigma}_{VV,t} = \sum_{i=0}^{2k} w_i (\overline{\chi}_{t,i} - \overline{V}_t)(\overline{\chi}_{t,i} - \overline{V}_t)^\top + Q_{t-1}.
  \]

- Re-generate sigma points $\tilde{\chi}_t$ based on the forecasted state mean and covariance $\overline{V}_t$ & $\overline{\Sigma}_{VV,t}$.

- Compute the forecasted mean and covariances of the measurements,
  \[
  \overline{\xi}_{t,i} = h(\tilde{\chi}_{t,i}; \Theta), \quad \overline{y}_t = \sum_{i=0}^{2k} w_i \overline{\xi}_{t,i},
  \]
  \[
  \overline{\Sigma}_{yy,t} = \sum_{i=0}^{2k} w_i (\overline{\xi}_{t,i} - \overline{y}_t)(\overline{\xi}_{t,i} - \overline{y}_t)^\top + R,
  \]
  \[
  \overline{\Sigma}_{Vy,t} = \sum_{i=0}^{2k} w_i (\tilde{\chi}_{t,i} - \overline{V}_t)(\overline{\xi}_{t,i} - \overline{y}_t)^\top.
  \]

- With the moment conditions, perform the filtering step the same as in Kalman filter.
### Pricing performance: Root mean squared error

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<td>120</td>
<td>4.014</td>
<td>1.081</td>
<td>1.057</td>
<td>0.984</td>
<td>0.714</td>
<td>0.561</td>
<td>0.572</td>
<td>0.735</td>
</tr>
</tbody>
</table>

Average rmse (in vol points) is 1.187 for benchmark and 0.83 for full model.
## Pricing performance: Explained variation

<table>
<thead>
<tr>
<th>$K/\tau$</th>
<th>1</th>
<th>3</th>
<th>6</th>
<th>12</th>
<th>24</th>
<th>36</th>
<th>48</th>
<th>60</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Reduced-form benchmark</strong></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>80</td>
<td>0.672</td>
<td>0.904</td>
<td>0.919</td>
<td>0.932</td>
<td>0.948</td>
<td>0.960</td>
<td>0.954</td>
<td>0.931</td>
</tr>
<tr>
<td>90</td>
<td>0.869</td>
<td>0.962</td>
<td>0.972</td>
<td>0.975</td>
<td>0.981</td>
<td>0.982</td>
<td>0.973</td>
<td>0.951</td>
</tr>
<tr>
<td>100</td>
<td>0.957</td>
<td>0.988</td>
<td>0.983</td>
<td>0.981</td>
<td>0.990</td>
<td>0.991</td>
<td>0.981</td>
<td>0.962</td>
</tr>
<tr>
<td>110</td>
<td>0.795</td>
<td>0.966</td>
<td>0.949</td>
<td>0.949</td>
<td>0.978</td>
<td>0.988</td>
<td>0.983</td>
<td>0.968</td>
</tr>
<tr>
<td>120</td>
<td>0.289</td>
<td>0.878</td>
<td>0.907</td>
<td>0.904</td>
<td>0.947</td>
<td>0.976</td>
<td>0.980</td>
<td>0.969</td>
</tr>
<tr>
<td><strong>Average</strong></td>
<td>0.717</td>
<td>0.940</td>
<td>0.946</td>
<td>0.948</td>
<td>0.969</td>
<td>0.979</td>
<td>0.974</td>
<td>0.956</td>
</tr>
<tr>
<td><strong>Full model</strong></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>80</td>
<td>0.897</td>
<td>0.972</td>
<td>0.961</td>
<td>0.970</td>
<td>0.983</td>
<td>0.987</td>
<td>0.971</td>
<td>0.936</td>
</tr>
<tr>
<td>90</td>
<td>0.965</td>
<td>0.985</td>
<td>0.982</td>
<td>0.982</td>
<td>0.990</td>
<td>0.996</td>
<td>0.989</td>
<td>0.965</td>
</tr>
<tr>
<td>100</td>
<td>0.968</td>
<td>0.994</td>
<td>0.991</td>
<td>0.986</td>
<td>0.991</td>
<td>0.996</td>
<td>0.992</td>
<td>0.977</td>
</tr>
<tr>
<td>110</td>
<td>0.930</td>
<td>0.979</td>
<td>0.977</td>
<td>0.974</td>
<td>0.984</td>
<td>0.989</td>
<td>0.989</td>
<td>0.978</td>
</tr>
<tr>
<td>120</td>
<td>0.293</td>
<td>0.938</td>
<td>0.939</td>
<td>0.945</td>
<td>0.971</td>
<td>0.981</td>
<td>0.981</td>
<td>0.973</td>
</tr>
<tr>
<td><strong>Average</strong></td>
<td>0.810</td>
<td>0.974</td>
<td>0.970</td>
<td>0.971</td>
<td>0.984</td>
<td>0.990</td>
<td>0.984</td>
<td>0.966</td>
</tr>
</tbody>
</table>

Likelihood: 38,265 for benchmark and 46,651 for full model.
Relative variance and skew contributions

<table>
<thead>
<tr>
<th>$\Theta$</th>
<th>Estimates</th>
<th>Std Error</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\rho$</td>
<td>2.8427</td>
<td>0.0074</td>
</tr>
<tr>
<td>$\rho$</td>
<td>-0.8354</td>
<td>0.0016</td>
</tr>
<tr>
<td>$\sigma_J$</td>
<td>5.6355</td>
<td>0.0272</td>
</tr>
<tr>
<td>$\nu_{J+}$</td>
<td>0.0000</td>
<td>0.0000</td>
</tr>
<tr>
<td>$\nu_{J-}$</td>
<td>0.1926</td>
<td>0.0002</td>
</tr>
</tbody>
</table>

- Average instantaneous return variance contributions from $(X_t, \nu_t^Z, \nu_t^J)$:
  
  \[ E_P[X_t^{−2\rho}] = 0.0119(10.91\%), \quad \text{variance (vol)} \]
  
  \[ E_P[\nu_t^Z] = 0.0231(15.19\%), \]
  
  \[ E_P[(\nu_{J+}^2 + \nu_{J−}^2)\nu_t^J] = 0.0116(10.79\%). \]

- All three types of interactions are strong: Leverage effect ($\rho = 2.8427$), volatility feedback ($\rho = -0.8354$), and self-excitement ($\sigma_J = 5.6355$).

- Much larger downside jumps than upside jumps ($\nu_{J−} \geq \nu_{J+}$).
Different term structure effects

<table>
<thead>
<tr>
<th>Θ</th>
<th>Estimates</th>
<th>Std Error</th>
</tr>
</thead>
<tbody>
<tr>
<td>( p )</td>
<td>2.8427</td>
<td>0.0074</td>
</tr>
<tr>
<td>( \kappa_Z )</td>
<td>3.0114</td>
<td>0.0127</td>
</tr>
<tr>
<td>( \kappa_J )</td>
<td>0.0009</td>
<td>0.0000</td>
</tr>
</tbody>
</table>

Different risk-neutral dynamics lead to different term structure responses:

- \( \nu^X_t \): Mean-repelling (drift=\( p(2p + 1)(\nu^X_t)^2 dt \)). Responses to shocks become larger at longer maturities.
- \( \nu^Z_t \): Strong mean reversion (\( \kappa_Z = 3.0114 \)). Responses decline quickly as option maturity increases.
- \( \nu^J_t \): Slow mean reversion (\( \kappa_J = 0.0009 \)). Responses do not decline.

⇒ The impacts are \( \nu^Z_t \) are mainly at short term options. \( X_t \) and \( \nu^J_t \) extend to long-term options.
The term structure of at-the-money implied volatility

- Solid lines: Evaluated at the sample average of \((X_t, \nu^Z_t, \nu^J_t)\).
- Dashed lines: Evaluated at 90th-percentile for one state variable.
- Dashed lines: Evaluated at 10th-percentile for one state variable.
The capital structure decision

\[
dX_t = X_t^{1-p} \left( a_X - \kappa_{XX} X_t - \kappa_{XZ} v_t^Z - \kappa_{XJ} v_t^J \right) dt + X_t^{1-p} dW_t^P
\]

<table>
<thead>
<tr>
<th>( \Theta )</th>
<th>Estimates</th>
<th>Std Error</th>
</tr>
</thead>
<tbody>
<tr>
<td>( a_X )</td>
<td>0.0003</td>
<td>0.0000</td>
</tr>
<tr>
<td>( \kappa_{XX} )</td>
<td>0.0001</td>
<td>0.0000</td>
</tr>
<tr>
<td>( \kappa_{XZ} )</td>
<td>17.5360</td>
<td>0.3087</td>
</tr>
<tr>
<td>( \kappa_{XJ} )</td>
<td>-0.0774</td>
<td>0.0000</td>
</tr>
</tbody>
</table>

- \( \kappa_{XX} = 0.0001 \): Capital structure is very persistent.
- \( \kappa_{XZ} = 17.536 \): High \textit{diffusion} business risk reduces \( X_t \) and hence increases the financial leverage.
- \( \kappa_{XJ} = -0.0774 \): High \textit{jump} business risk increases \( X_t \) and hence reduces the financial leverage.
## Market prices of risks

<table>
<thead>
<tr>
<th>θ</th>
<th>Estimates</th>
<th>Std Error</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\gamma^v$</td>
<td>-17.4507</td>
<td>0.3048</td>
</tr>
<tr>
<td>$\gamma^j$</td>
<td>0.4468</td>
<td>0.0023</td>
</tr>
</tbody>
</table>

(i) Market price of diffusion variance risk ($\gamma^v$) is highly negative → Negative variance risk premium.

Market price of return jump risk ($\gamma^v$) is positive:

- Risk-neutral return innovation distribution is more *negatively* skewed than the statistical distribution:
  \[
  v^P_{J+} = \frac{v_{J+}}{1 - \gamma^j v_{J+}} > v_{J+},
  \]
  \[
  v^P_{J-} = \frac{v_{J-}}{1 + \gamma^j v_{J-}} < v_{J-}.
  \]

(ii) Instantaneous variance contribution from jumps is larger under the risk-neutral measure than under the statistical measure:
  \[
  (v^2_{J+} + v^2_{J-}) v^J_t > \left((v^P_{J+})^2 + (v^P_{J-})^2\right)v^J_t.
  \]

(iii) Negative risk premium on the jump arrival rate ($v^J_t$):
  \[
  \sigma_j (v^P_{J-} - v_{J-}) v^J_t < 0.
  \]

(i), (ii), & (iii) ⇒ Negative index return variance risk premium.
The risk contribution from financial leverage ($v_t^X$) reached historical highs before the burst of the Nasdaq bubble.

The diffusion business risk ($v_t^Z$) peaked during the 2003 recession.

The jump risk ($v_t^J$) reached its highest level during the Asian crises.
Leverage increases in good times.

Diffusion risk and jump risk both increase when market is down.

Some predictions?
Multiple sources of implied volatility skew

Our model embeds 4 mechanisms in generating the implied volatility skew: (1) leverage effect ($p > 0$), (2) volatility feedback ($\rho < 0$), (3) crash risk ($v_{J-} \gg v_{J+}$), (4) self-excitement of crash events ($\sigma_{J}$).

$$SK_{t,T} = (IV_{t,T}(80\%) - IV_{t,T}(120%))/(|d_{t,T}(80\%) - d_{t,T}(120%)|)$$
Concluding remarks

- Equity return and volatility interact through several distinct channels.

- It is helpful to separately model the variations of the financial leverage and the business risk
  - to bridge the gaps in the literature,
  - to disentangle the different mechanisms of interaction, and
  - to generate good pricing performance on equity options over both short and long option maturities.

- The approach has potentials in analyzing single name stocks (options).