Liuren Wu at Baruch College

The talk is based on joint work with Peter Carr and Markus Leippold

QWAFAFEW
Wednesday, August 19th, 2009
Overview

- Variance is often used as a risk measure for financial security returns.
- But variance itself is uncertain.
- When investing in an asset, an investor faces at least two sources of risk: (1) return, and (2) return variance.
- Many variance-related derivative products have been developed both over the counter and in listed markets.
- One of the most actively traded and simplest OTC variance contract is a variance swap:
  - The contract has zero value at inception.
  - At maturity, the long side of the variance swap contract receives the difference between a standard measure of the realized variance and a fixed rate, called the variance swap rate, determined at the inception of the contract.
- This talk discusses three variance risk related questions:
  - How does the market price variance risk?
  - How should we invest in variance swap contracts?
  - How does variance risk/premium interact with equity risk premium?
Measuring variance risk premium through variance swaps

The literature: model estimation, returns on delta-hedged option positions, stock market portfolios...

We propose to measure the variance risk premium through YTM on a variance swap investment.

The two counter-parties in a variance swap contract agree to pay/receive the difference between the realized variance and a fixed variance swap rate over a certain horizon.

$\text{Fixed Swap Rate (VS)} \iff \text{Realized Variance (RV)}$

No-arbitrage dictates that $VS_{t,T} = E^{Q}_{t}[RV_{t,T}]$.

Define variance (swap) risk premium:

$VRP_{t,T} \equiv E^{P}_{t}[RV_{t,T}] - E^{Q}_{t}[RV_{t,T}] = E^{P}_{t}[RV_{t,T}] - VS_{t,T}.$

The average variance risk premium is simply the average return on fixed notional investment in variance swap contracts:

$RP_{t,T} = RV_{t,T} - VS_{t,T}.$
In the absence of variance swap, go to the log profile

- While VS quotes may not be readily available, vanilla options have been actively traded on listed markets for several decades.

- We can use vanilla options to create a “log profile” that approximates the variance swap rate.

- A Taylor expansion with remainder of $\ln F_T$ about the point $F_t$ implies:

  $$
  \ln F_T = \ln F_t + \frac{1}{F_t} (F_T - F_t) - \int_0^{F_t} \frac{1}{K^2} (K - F_T)^+ dK - \int_{F_t}^\infty \frac{1}{K^2} (F_T - K)^+ dK.
  $$

- Thus, the forward value of the particular portfolio of vanilla options is equal to the negative of the risk-neutral expected value of the log return,

  $$
  \frac{2}{T - t} \int_0^\infty \frac{1}{K^2} O_t(K, T) dK = \frac{-2}{T - t} \mathbb{E}_t^Q [\ln F_T/F_t] \equiv LP_{t,T},
  $$

  where $O_t(K, T)$ denotes the forward value of an OTM option at strike $K$ and maturity $T$.

- We refer to this portfolio of options as the log profile.
Replicating a variance swap contract with vanilla options

Under *purely continuous dynamics*, the annualized realized variance can be replicated by dynamic trading in futures and static positions in options,

\[
RV_{t, T} = \frac{2}{T - t} \int_t^T \left[ \frac{1}{F_s} - \frac{1}{F_t} \right] dF_s \\
+ \frac{2}{T - t} \left[ \int_0^{F_t} \frac{1}{K^2} (K - F_T)^+ dK + \int_{F_t}^{\infty} \frac{1}{K^2} (F_T - K)^+ dK \right].
\]

- The risk-neutral expected profits from the futures trading is zero.
- The risk-neutral expected value of the return variance is equal to the forward value of a portfolio of vanilla options,

\[
\mathbb{E}_t^Q [RV_{t, T}] = \frac{2}{T - t} \int_0^{\infty} \frac{1}{K^2} O_t(K, T) dK
\]

- Thus, under continuous price dynamics, the variance swap rate \((VS_{t, T})\) is equal to the log profile \((LP_{t, T})\):

\[
VS_{t, T} = \mathbb{E}_t^Q [RV_{t, T}] = \frac{-2}{T - t} \mathbb{E}_t^Q [\ln F_T/F_t] = LP_{t, T}.
\]
In the presence of price jumps, the variance swap rate ($VS_{t,T}$) and the log profile ($LP_{t,T}$) differ due to third and higher order powers of $d \ln F_s$,

$$VS_{t,T} = LP_{t,P} + \varepsilon,$$

with

$$\varepsilon = \frac{-2}{T - t} \mathbb{E}_t^Q \int_t^T \int_{\mathbb{R}^0} \left[ e^x - 1 - x - \frac{x^2}{2} \right] \nu_s(x) dx ds.$$

Numerical analysis shows that the error term $\varepsilon$ is small under commonly specified dynamics and parameters for equity indexes.

- Variance swap rate is higher than the log profile when there are more down jumps than up jumps.

- Single names are a bit complicated...
Evidence on SPX

- Variance swap rate quotes are obtained from a broker dealer at 2, 3, 6, 12, 24 months. 1996 to now.

- Log profiles are constructed from SPX option quotes.
  - Different methods to interpolate across the strike dimension does not dramatically change the conclusions, esp. at short maturities.

- Compute the logarithm of the proportionality coefficient $\ln\left(\frac{LP_t, T}{VS_t, T}\right)$ at different maturities.

The log profile and the variance swap rates are very close to each other.
Evidence on variance risk premium

- Seven plus years of options data from OptionMetrics.

- At each day, we construct
  - A 30-day log profile to approximate the 30-day variance swap rate,
  - A 30-day realized variance based on daily returns for 5 indexes, 35 individual stocks.

- We look at the time series behavior of:
  - $RP = (RV - VS) \times 100$: Return on $100 notational from long the 30-day variance swap contract and holding it to maturity.
  - $LRP = \ln (RV/VS)$: The excess log return, with $VS$ rate as the current forward and $RV$ the future spot.
### Variance risk premiums on stock indexes and single names

<table>
<thead>
<tr>
<th>Ticker</th>
<th>A: ((RV - VS) \times 100)</th>
<th>B: (\ln \left(\frac{RV}{VS}\right))</th>
<th>IR</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Mean</td>
<td>Std</td>
<td>(t)</td>
</tr>
<tr>
<td>SPX</td>
<td>-2.74</td>
<td>3.63</td>
<td>-8.39</td>
</tr>
<tr>
<td>OEX</td>
<td>-2.36</td>
<td>3.57</td>
<td>-7.02</td>
</tr>
<tr>
<td>DJX</td>
<td>-2.58</td>
<td>3.86</td>
<td>-6.37</td>
</tr>
<tr>
<td>NDX</td>
<td>-2.43</td>
<td>10.24</td>
<td>-2.54</td>
</tr>
<tr>
<td>MSFT</td>
<td>-3.20</td>
<td>12.31</td>
<td>-3.32</td>
</tr>
<tr>
<td>INTC</td>
<td>2.49</td>
<td>19.07</td>
<td>1.34</td>
</tr>
<tr>
<td>IBM</td>
<td>-1.68</td>
<td>10.24</td>
<td>-1.80</td>
</tr>
<tr>
<td>AMER</td>
<td>-3.51</td>
<td>23.76</td>
<td>-2.05</td>
</tr>
<tr>
<td>DELL</td>
<td>-4.43</td>
<td>21.35</td>
<td>-2.15</td>
</tr>
<tr>
<td>CSCO</td>
<td>-2.30</td>
<td>20.31</td>
<td>-1.42</td>
</tr>
<tr>
<td>GE</td>
<td>-2.24</td>
<td>7.63</td>
<td>-3.52</td>
</tr>
</tbody>
</table>

- Shorting variance swap on equity indexes make money on average.
- Variance risk premiums on single names show large cross-sectional variations — more work is needed on single names.
Expectation hypothesis regressions on SPX

- **EH1**: Constant variance risk premium $RP$ — rejected.
  
  \[
  RV_{t,T} = 0.010 + 0.455 \, VS_{t,T} + e, \quad R^2 = 26.2\%, \quad (1.42) \quad (\text{−4.60})
  \]

- **EH2**: Constant log variance risk premium $LRP$ — not rejected.
  
  \[
  \ln RV_{t,T} = -0.891 + 0.919 \, \ln VS_{t,T} + e, \quad R^2 = 37.8\%, \quad (−2.59) \quad (−0.68)
  \]

Caveat: Transaction cost will eat into the return more during low vol periods.
**PL from shorting SPX variance swaps**

- The negative variance risk premium on stock indexes suggests that shorting variance swap generates positive average returns.
- CBOE’s VIX approximates the log profile of SPX, which we can use as an approximate quote for 30-day variance swap rate.
- Each day, short $1 million notional of 30-day variance swap on SPX and holding the short position to maturity.

The PL from 1990 to 2007: Mean=$1.39k, Std=$2.17k/contract. IR= 2.2

Comparison: IR from SPX: 0.4
until Oct. 2008, when short-variance funds are wiped out due to leverage
But investing in SPX is even worse
"Optimal variance swap investments," JFQA, with Markus Leippold.

- Analyzing variance swap rates across different maturities (2m to 24m):
  \[ \Rightarrow \] Two stochastic volatility factors: one controlling the short-term variance \((v_t)\), the other controlling the long-term tendency \((m_t)\).

- Consider a CRRA investor who puts a fraction of her wealth \((w)\) in the equity index and fractions of her wealth \((n_1, n_2)\) as notional in two variance swap contracts to span the two variance risk factors and to maximize her terminal wealth.

- Under affine structures and proportional risk premiums, the allocation weights \((w, n_1, n_2)\) are constant over time:

\[
\begin{align*}
w_t &= \frac{1}{\eta} \left( \gamma S - \frac{\rho}{\sqrt{1-\rho^2}} \gamma^z \right), \\
n_{1t} &= \frac{1}{\eta D} \left[ \left( \frac{\gamma^z}{\sigma_v \sqrt{1-\rho^2}} + h_v(u) \right) \phi_m(T_2 - t) - \left( \frac{\gamma^m}{\sigma_m} + h_m(u) \right) \phi_v(T_2 - t) \right], \\
n_{2t} &= \frac{1}{\eta D} \left[ - \left( \frac{\gamma^z}{\sigma_v \sqrt{1-\rho^2}} + h_v(u) \right) \phi_m(T_1 - t) + \left( \frac{\gamma^m}{\sigma_m} + h_m(u) \right) \phi_v(T_1 - t) \right].
\end{align*}
\]
In-sample investment analysis

  - With a relative risk aversion of 200, we have \( w = -0.121, \ n_1 = -0.873 \) for 2m VS, \( n_2 = 0.195 \) for 2y VS.
  - The high risk aversion is to counter to the extremely high variance risk premium estimate so that we can keep leverage in check.

Out-of-sample IR = 2.3 vs IR for SPX alone = 0.54.
Out-of-sample investment analysis

Liuren Wu  Variance Risk Premia
Out-of-sample investment analysis

Cumulative wealth

IR = 0.40 (blue, fixed weights),
    1.84 (green, weights varying with term structure information),
    −0.12 (red, stock only).

Liuren Wu
Variance Risk Premia
Summary: What we have learned

- Comparing variance swap rate with ex post realized variance presents a simple, direct way of measuring variance risk premium.

- The log profile matches the variance swap quote fairly well for SPX.

- Variance risk on equity index is highly priced.
  - Shorting variance swaps generates positive PL on average.

- The magnitude of the variance risk premium is proportional to the variance level.

- A portfolio that is short short-term index variance swap, long long-term variance swap, and short the equity index performs better than long the equity index alone,
  - even during the current financial meltdown.
What we still do not know

- What are the sources of the equity index variance risk premium?
  - Jump in return (even when variance is constant).
  - Return risk premium and return-variance correlation.
  - Risk premiums on independent variance risk.

  What’s the relative contribution of each component?

- How to measure variance risk premiums (and their sources) on single names?
  - The possibility of default makes variance on log returns undefined.
  - Truncated payoffs in practical contracts.
  - Log profile can differ significantly from variance swap rates.
  - Default risk (premium) plays a big role.

- How to relate single name variance risk to market variance risk (in a tractable and yet reasonable way)?

- What’s the guideline for variance investments across names?
From variance risk to return risk premium

- How does our better understanding/measurement of variance risk/premium affect our stock return risk premium determination?

- Commonly identified return risk factor: the market portfolio return (CAPM), size, book-to-market, momentum.

- Variance risk seems to be another factor:
  - Construct one-month at-the-money implied variance for each stock.
  - Use the log changes in the implied variance as a proxy for variance (option) investment return.
  - Construct an equal weighted market variance portfolio return.
  - Regress each variance series on this market portfolio.
  - High variance beta $\rightarrow$ low return on stock portfolios.

- Questions: How informative is the implied volatility surface about future stock/option returns?