Variance Risk Premia

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The talk is based on joint work with Peter Carr and Markus Leippold

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Overview

- Variance is often used as a risk measure for financial security returns.
- But variance itself is uncertain.
- When investing in an asset, an investor faces at least two sources of risk: (1) return, and (2) return variance.
- Many variance-related derivative products have been developed both over the counter and in listed markets.
- One of the most actively traded and simplest OTC variance contract is a variance swap:
  - The contract has zero value at inception.
  - At maturity, the long side of the variance swap contract receives the difference between a standard measure of the realized variance and a fixed rate, called the variance swap rate, determined at the inception of the contract.
- This talk discusses two related questions (in the equity market):
  - How does the market price variance risk?
  - How should we invest in variance swap contracts?
Measuring variance risk premium through variance swaps

- The literature: model estimation, returns on delta-hedged option positions, stock market portfolios...

- We propose to measure the variance risk premium through YTM on a variance swap investment.
  
  - The two counter-parties in a variance swap contract agree to pay/receive the difference between the realized variance and a fixed variance swap rate over a certain horizon.

\[
\text{Fixed Swap Rate (}VS\text{)} \iff \text{Realized Variance (}RV\text{)}
\]

- No-arbitrage dictates that \(VS_{t,T} = \mathbb{E}^{Q}_{t}[RV_{t,T}]\).

- Define variance (swap) risk premium:
  \[
  VRP_{t,T} \equiv \mathbb{E}^{P}_{t}[RV_{t,T}] - \mathbb{E}^{Q}_{t}[RV_{t,T}] = \mathbb{E}^{P}_{t}[RV_{t,T}] - VS_{t,T}.
  \]

- The average variance risk premium is simply the average return on fixed notional investment in variance swap contracts:
  \[
  RP_{t,T} = RV_{t,T} - VS_{t,T}.
  \]
In the absence of variance swap, go to the log profile

- While VS quotes may not be readily available, vanilla options have been actively traded on listed markets for several decades.

- We can use vanilla options to create a “log profile” that approximates the variance swap rate.

- A Taylor expansion with remainder of \( \ln F_T \) about the point \( F_t \) implies:

\[
\ln F_T = \ln F_t + \frac{1}{F_t} (F_T - F_t) - \int_0^{F_t} \frac{1}{K^2} (K - F_T)^+ dK - \int_{F_t}^{\infty} \frac{1}{K^2} (F_T - K)^+ dK.
\]

- Thus, the forward value of the particular portfolio of vanilla options is equal to the negative of the risk-neutral expected value of the log return,

\[
\frac{2}{T - t} \int_0^{\infty} \frac{1}{K^2} O_t(K, T) dK = \frac{-2}{T - t} \mathbb{E}^Q_t [\ln F_T / F_t] \equiv LP_{t, T},
\]

where \( O_t(K, T) \) denotes the forward value of an OTM option at strike \( K \) and maturity \( T \).

- We refer to this portfolio of options as the log profile.
Replicating a variance swap contract with vanilla options

Under *purely continuous dynamics*, the annualized realized variance can be replicated by dynamic trading in futures and static positions in options,

\[
RV_{t,T} = \frac{2}{T-t} \int_t^T \left[ \frac{1}{F_s} - \frac{1}{F_t} \right] dF_s \\
+ \frac{2}{T-t} \left[ \int_0^{F_t} \frac{1}{K^2} (K - F_T)^+ dK + \int_{F_t}^\infty \frac{1}{K^2} (F_T - K)^+ dK \right].
\]

- The risk-neutral expected profits from the futures trading is zero.
- The risk-neutral expected value of the return variance is equal to the forward value of a portfolio of vanilla options,

\[
\mathbb{E}_t^Q [RV_{t,T}] = \frac{2}{T-t} \int_0^\infty \frac{1}{K^2} O_t(K, T) dK
\]

- Thus, under continuous price dynamics, the variance swap rate \((VS_{t,T})\) is equal to the log profile \((LP_{t,T})\):

\[
VS_{t,T} \equiv \mathbb{E}_t^Q [RV_{t,T}] = \frac{-2}{T-t} \mathbb{E}_t^Q [\ln F_T / F_t] \equiv LP_{t,T}.
\]
In the presence of price jumps, the variance swap rate \( (VS_{t,T}) \) and the log profile \( (LP_{t,T}) \) differ due to third and higher order powers of \( d \ln F_s \),

\[
VS_{t,T} = LP_{t,P} + \varepsilon,
\]

with

\[
\varepsilon = \frac{-2}{T - t} \mathbb{E}_t^Q \int_t^T \int_{\mathbb{R}^0} \left[ e^x - 1 - x - \frac{x^2}{2} \right] \nu_s(x) dx ds.
\]

Numerical analysis shows that the error term \( \varepsilon \) is small under commonly specified dynamics and parameters for equity indexes.

Single names are a bit complicated...
When return is driven by a time-changed Lévy process

\[ \ln(F_T/F_t) = X_{t,T} - \kappa_X(1)T_{t,T}, \]

- \( X_t \) denotes a Lévy martingale process with \( \mathbb{E}^Q[X_t] = 0 \), and \( T_{t,T} \) denotes the stochastic time change, \( T_{t,T} \equiv \int_t^T \nu_s ds \).
- \( \kappa_X(s) \) is the cumulant exponent of \( X \), \( \kappa_X(s) \equiv \frac{1}{t} \ln \mathbb{E}^Q[e^{sX_t}] \).
- The log profile is
  \[ LP_{t,T} \equiv \frac{-2}{T-t} \mathbb{E}^Q_t [\ln F_T/F_t] = 2\kappa_X(1) \mathbb{E}^Q_t [T_{t,T}]/(T-t). \]
- The variance swap is
  \[ VS_{t,T} = \mathbb{E}^Q_t [RV_{t,T}] = \kappa''_X(0) \mathbb{E}^Q_t [T_{t,T}]/(T-t). \]
- Let \( A_{t,T} = \mathbb{E}^Q_t [T_{t,T}/(T-t)] \) denote the expected value of the annualized time change, we have
  \[ LP_{t,T} = 2\kappa_X(1) A_{t,T}, \quad VS_{t,T} = \kappa''_X(0) A_{t,T}, \]
- Hence, \( LP_{t,T} = \beta_X VS_{t,T} \), with the proportionality coeff \( \beta_X = \frac{2\kappa_X(1)}{\kappa''_X(0)} \).
- The ratio does not depend on the volatility dynamics, only on the return innovation (Lévy process).
- If \( X_t = W_t \), \( \kappa_X(s) = \frac{1}{2} s^2 \), \( \beta_X = 1. \Rightarrow LP = VS. \)
Merton (76) jump-diffusion

Dampened power law

Mean jump size

Mean up/down jump size ratio

Variance swap rate is higher than the log profile ($\beta_X < 1$) when there are more down jumps than up jumps.
Variance swap rate quotes are obtained from a broker dealer at 2, 3, 6, 12, 24 months. 1996 to now.

Log profiles are constructed from SPX option quotes.

Different methods to interpolate across the strike dimension does not dramatically change the conclusions, esp. at short maturities.

Compute the logarithm of the proportionality coefficient
\[ \ln \left( \frac{LP_{t,T}}{VS_{t,T}} \right) \]

at different maturities.

The log profile and the variance swap rates are very close to each other.
“Variance risk premiums,” RFS, with Peter Carr.

- Seven plus years of options data from OptionMetrics.

- At each day, we construct
  - A 30-day log profile to approximate the 30-day variance swap rate,
  - A 30-day realized variance based on daily returns

  for 5 indexes, 35 individual stocks.

- We look at the time series behavior of:
  - $RP = (RV - VS) \times 100$: Return on $100 notational from long the 30-day variance swap contract and holding it to maturity.
  - $LRP = \ln (RV/VS)$: The excess log return, with $VS$ rate as the current forward and $RV$ the future spot.
## Variance risk premiums on stock indexes and single names

<table>
<thead>
<tr>
<th>Ticker</th>
<th>A: $\left( RV - VS \right) \times 100$</th>
<th>B: $\ln \left( \frac{RV}{VS} \right)$</th>
<th>IR</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Mean</td>
<td>Std</td>
<td>$t$</td>
</tr>
<tr>
<td>SPX</td>
<td>-2.74</td>
<td>3.63</td>
<td>-8.39</td>
</tr>
<tr>
<td>OEX</td>
<td>-2.36</td>
<td>3.57</td>
<td>-7.02</td>
</tr>
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<td>DJX</td>
<td>-2.58</td>
<td>3.86</td>
<td>-6.37</td>
</tr>
<tr>
<td>NDX</td>
<td>-2.43</td>
<td>10.24</td>
<td>-2.54</td>
</tr>
<tr>
<td>MSFT</td>
<td>-3.20</td>
<td>12.31</td>
<td>-3.32</td>
</tr>
<tr>
<td>INTC</td>
<td>2.49</td>
<td>19.07</td>
<td>1.34</td>
</tr>
<tr>
<td>IBM</td>
<td>-1.68</td>
<td>10.24</td>
<td>-1.80</td>
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<tr>
<td>AMER</td>
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<td>23.76</td>
<td>-2.05</td>
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<td>21.35</td>
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</tr>
<tr>
<td>CSCO</td>
<td>-2.30</td>
<td>20.31</td>
<td>-1.42</td>
</tr>
<tr>
<td>GE</td>
<td>-2.24</td>
<td>7.63</td>
<td>-3.52</td>
</tr>
</tbody>
</table>

- Shorting variance swap on equity indexes make money on average.
- Variance risk premiums on single names show large cross-sectional variations — more work is needed on single names.
Expectation hypothesis regressions on SPX

- **EH1**: Constant variance risk premium $RP$ — rejected.
  
  $$RV_{t,T} = 0.010 + 0.455 \, VS_{t,T} + e, \quad R^2 = 26.2\%,$$
  
  (1.42) $\quad$ (-4.60)

- **EH2**: Constant log variance risk premium $LRP$ — not rejected.
  
  $$\ln RV_{t,T} = -0.891 + 0.919 \, \ln VS_{t,T} + e, \quad R^2 = 37.8\%,$$
  
  (-2.59) $\quad$ (-0.68)

Caveat: Transaction cost will eat into the return more during low vol periods.
The negative variance risk premium on stock indexes suggests that shorting variance swap generates positive average returns.

CBOE’s VIX approximates the log profile of SPX, which we can use as an approximate quote for 30-day variance swap rate.

Each day, short $1 million notional of 30-day variance swap on SPX and holding the short position to maturity.

The PL from 1990 to 2007: Mean=$1.39k, Std=$2.17k/contract. IR= 2.2

Comparison: IR from SPX: 0.4
PL from shorting SPX variance swaps

until Oct. 2008, when short-variance funds are wiped out due to leverage
But investing in SPX is even worse
"Optimal variance swap investments," JFQA, with Markus Leippold.

- Analyzing variance swap rates across different maturities (2m to 24m):
  - Two stochastic volatility factors: one controlling the short-term variance ($\nu_t$), the other controlling the long-term tendency ($m_t$).
- Consider a CRRA investor who puts a fraction of her wealth ($w$) in the equity index and fractions of her wealth ($n_1, n_2$) as notional in two variance swap contracts to span the two variance risk factors and to maximize her terminal wealth.
- Under affine structures and proportional risk premiums, the allocation weights ($w, n_1, n_2$) are constant over time:

$$w_t = \frac{1}{\eta} \left( \gamma S - \frac{\rho}{\sqrt{1-\rho^2}} \gamma^z \right),$$

$$n_{1t} = \frac{1}{\eta D} \left[ \left( \frac{\gamma^z}{\sigma_v \sqrt{1-\rho^2}} + h_v(u) \right) \phi_m(T_2 - t) - \left( \frac{\gamma^m}{\sigma_m} + h_m(u) \right) \phi_v(T_2 - t) \right],$$

$$n_{2t} = \frac{1}{\eta D} \left[ - \left( \frac{\gamma^z}{\sigma_v \sqrt{1-\rho^2}} + h_v(u) \right) \phi_m(T_1 - t) + \left( \frac{\gamma^m}{\sigma_m} + h_m(u) \right) \phi_v(T_1 - t) \right].$$
In-sample investment analysis

  - With a relative risk aversion of 200, we have $w = -0.121$, $n_1 = -0.873$ for 2m VS, $n_2 = 0.195$ for 2y VS.
  - The high risk aversion is to counter to the extremely high variance risk premium estimate so that we can keep leverage in check.

Out-of-sample IR = 2.3 vs IR for SPX alone = 0.54.
Out-of-sample IR = 0.33 vs IR for SPX alone = -0.08.
Comparing variance swap rate with ex post realized variance presents a simple, direct way of measuring variance risk premium.

The log profile matches the variance swap quote fairly well for SPX.

Variance risk on equity index is highly priced.

- Shorting variance swaps generates positive PL on average.

The magnitude of the variance risk premium is proportional to the variance level.

A portfolio that is short short-term index variance swap, long long-term variance swap, and short the equity index performs better than long the equity index alone,

- even during the current financial meltdown.
What we still do not know

- What are the sources of the equity index variance risk premium?
  - Jump in return (even when variance is constant).
  - Return risk premium and return-variance correlation.
  - Risk premiums on independent variance risk.

  What’s the relative contribution of each component?

- How to measure variance risk premiums (and their sources) on single names?
  - The possibility of default makes variance on log returns undefined.
  - Truncated payoffs in practical contracts.
  - Log profile can differ significantly from variance swap rates.
  - Default risk (premium) plays a big role.

- How to relate single name variance risk to market variance risk (in a tractable and yet reasonable way)?

- What’s the guideline for variance investments across names?