Variance Risk Premia

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The talk is based on joint work with Peter Carr and Markus Leippold

Conference on *Econometric Modeling in Risk Management*
University of Waterloo, March 27, 2009

Overview

- Variance is often used as a risk measure for financial security returns.
- But variance itself is uncertain.
- When investing in an asset, an investor faces at least two sources of risk: (1) return, and (2) return variance.
- Many variance-related derivative products have been developed both over the counter and in listed markets.
- One of the most actively traded and simplest OTC variance contract is a *variance swap*:
  - The contract has zero value at inception.
  - At maturity, the long side of the variance swap contract receives the difference between a standard measure of the realized variance and a fixed rate, called the *variance swap rate*, determined at the inception of the contract.
- This talk discusses two related questions (in the equity market):
  - How does the market price variance risk?
  - How should we invest in variance swap contracts?

Measuring variance risk premium through variance swaps

- The literature: model estimation, returns on delta-hedged option positions, stock market portfolios...
- We propose to measure the variance risk premium through YTM on a variance swap investment.
  - The two counter-parties in a variance swap contract agree to pay/receive the difference between the realized variance and a fixed variance swap rate over a certain horizon.

    Fixed Swap Rate (VS) $\rightleftharpoons$ **Realized Variance (RV)**

  - No-arbitrage dictates that $VS_{t,T} = \mathbb{E}_t^{\mathbb{Q}}[RV_{t,T}]$.
  - Define variance (swap) risk premium:
    $VRP_{t,T} \equiv \mathbb{E}_t^{\mathbb{P}}[RV_{t,T}] - \mathbb{E}_t^{\mathbb{Q}}[RV_{t,T}] = \mathbb{E}_t^{\mathbb{P}}[RV_{t,T}] - VS_{t,T}$.
  - The average variance risk premium is simply the average return on fixed notional investment in variance swap contracts:

$$RP_{t,T} = RV_{t,T} - VS_{t,T}.$$

In the absence of variance swap, go to the log profile

- While VS quotes may not be readily available, vanilla options have been actively traded on listed markets for several decades.
- We can use vanilla options to create a "log profile" that approximates the variance swap rate.
- A Taylor expansion with remainder of $\ln F_T$ about the point F_t implies:

$$\ln F_T = \ln F_t + \frac{1}{F_t}(F_T - F_t) - \int_0^{F_t} \frac{1}{K^2}(K - F_T)^+ dK - \int_{F_t}^{\infty} \frac{1}{K^2}(F_T - K)^+ dK.$$

- Thus, the forward value of the particular portfolio of vanilla options is equal to the negative of the risk-neutral expected value of the log return,

$$\frac{2}{T-t}\int_0^{\infty} \frac{1}{K^2} O_t(K,T) dK = \frac{-2}{T-t}\mathbb{E}_t^{\mathbb{Q}}\left[\ln F_T/F_t\right] \equiv LP_{t,T},$$

where $O_t(K,T)$ denotes the forward value of an OTM option at strike K and maturity T.

- We refer to this portfolio of options as the *log profile*.

Replicating a variance swap contract with vanilla options

Under *purely continuous dynamics*, the annualized realized variance can be replicated by dynamic trading in futures and static positions in options,

$$\begin{aligned} RV_{t,T} &= \tfrac{2}{T-t}\int_t^T \left[\tfrac{1}{F_s}-\tfrac{1}{F_t}\right] dF_s \\ &+\tfrac{2}{T-t}\left[\int_0^{F_t}\tfrac{1}{K^2}(K-F_T)^+dK+\int_{F_t}^{\infty}\tfrac{1}{K^2}(F_T-K)^+dK\right]. \end{aligned}$$

- The risk-neutral expected profits from the futures trading is zero.
- The risk-neutral expected value of the return variance is equal to the forward value of a portfolio of vanilla options,

$$\mathbb{E}_t^{\mathbb{Q}}\left[RV_{t,T}\right] = \frac{2}{T-t}\int_0^{\infty}\frac{1}{K^2}O_t(K,T)dK$$

- Thus, under continuous price dynamics, the variance swap rate ($VS_{t,T}$) is equal to the log profile ($LP_{t,T}$):

$$VS_{t,T} \equiv \mathbb{E}_t^{\mathbb{Q}}\left[RV_{t,T}\right] = \frac{-2}{T-t}\mathbb{E}_t^{\mathbb{Q}}\left[\ln F_T/F_t\right] \equiv LP_{t,T}.$$

Variance swaps and the log profile

- In the presence of price jumps, the variance swap rate ($VS_{t,T}$) and the log profile ($LP_{t,T}$) differ due to third and higher order powers of $d \ln F_s$,

$$VS_{t,T} = LP_{t,P} + \varepsilon,$$

  with
$$\varepsilon = \frac{-2}{T-t}\mathbb{E}_t^{\mathbb{Q}} \int_t^T \int_{\mathbb{R}^0} \left[e^x - 1 - x - \frac{x^2}{2} \right] \nu_s(x) dx ds.$$

- Numerical analysis shows that the error term ε is small under commonly specified dynamics and parameters for equity indexes.

- Single names are a bit complicated...

When return is driven by a time-changed Lévy process

$\ln(F_T/F_t) = X_{\mathcal{T}_{t,T}} - \kappa_X(1)\mathcal{T}_{t,T}$,

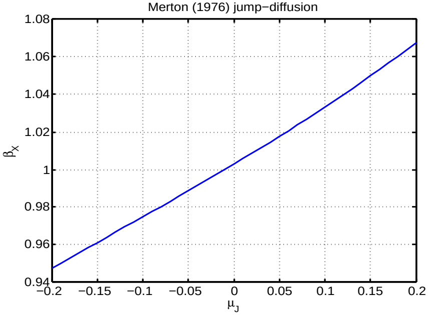
- X_t denotes a Lévy martingale process with $\mathbb{E}^{\mathbb{Q}}[X_t] = 0$, and $\mathcal{T}_{t,T}$ denotes the stochastic time change, $\mathcal{T}_{t,T} \equiv \int_t^T v_s ds$.
- $\kappa_X(s)$ is the cumulant exponent of X, $\kappa_X(s) \equiv \frac{1}{t} \ln \mathbb{E}^{\mathbb{Q}}\left[e^{sX_t}\right]$.
- The log profile is $LP_{t,T} \equiv \frac{-2}{T-t}\mathbb{E}_t^{\mathbb{Q}}[\ln F_T/F_t] = 2\kappa_X(1)\,\mathbb{E}_t^{\mathbb{Q}}[\mathcal{T}_{t,T}]/(T-t)$.
- The variance swap is $VS_{t,T} = \mathbb{E}_t^{\mathbb{Q}}[RV_{t,T}] = \kappa_X''(0)\,\mathbb{E}_t^{\mathbb{Q}}[\mathcal{T}_{t,T}]/(T-t)$.
- Let $\mathcal{A}_{t,T} = \mathbb{E}_t^{\mathbb{Q}}[\mathcal{T}_{t,T}/(T-t)]$ denote the expected value of the annualized time change, we have

$$LP_{t,T} = 2\kappa_X(1)\,\mathcal{A}_{t,T}, \quad VS_{t,T} = \kappa_X''(0)\,\mathcal{A}_{t,T},$$

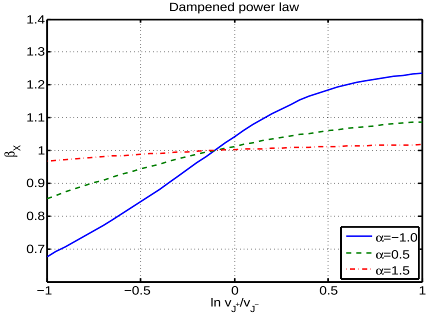
- Hence, $LP_{t,T} = \beta_X VS_{t,T}$, with the proportionality coeff $\beta_X = \frac{2\kappa_X(1)}{\kappa_X''(0)}$.
- The ratio does not depend on the volatility dynamics, only on the return innovation (Lévy process).
- If $X_t = W_t$, $\kappa_X(s) = \frac{1}{2}s^2$, $\beta_X = 1$. $\Rightarrow LP = VS$.

Lévy process examples

Merton (76) jump-diffusion	Dampened power law



Mean jump size

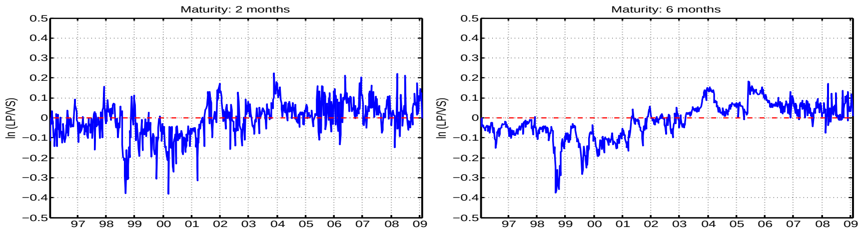


Mean up/down jump size ratio

Variance swap rate is higher than the log profile ($\beta_X < 1$) when there are more down jumps than up jumps.

Evidence on SPX

- Variance swap rate quotes are obtained from a broker dealer at 2,3,6,12,24 months. 1996 to now.
- Log profiles are constructed from SPX option quotes.
  - Different methods to interpolate across the strike dimension does not dramatically change the conclusions, esp. at short maturities.
- Compute the logarithm of the proportionality coefficient $\ln(LP_{t,T}/VS_{t,T})$ at different maturities.



The log profile and the the variance swap rates are very close to each other.

Evidence on vanilla options

"*Variance risk premiums*," RFS, with Peter Carr.

- Seven plus years of options data from OptionMetrics.
- At each day, we construct
  - A 30-day log profile to approximate the 30-day variance swap rate,
  - A 30-day realized variance based on daily returns

  for 5 indexes, 35 individual stocks.
- We look at the time series behavior of:
  - $RP = (RV - VS) \times 100$: Return on \$100 notational from long the 30-day variance swap contract and holding it to maturity.
  - $LRP = \ln(RV/VS)$: The excess log return, with VS rate as the current forward and RV the future spot.

Variance risk premiums on stock indexes and single names

Ticker	A: $(RV - VS) \times 100$			B: $\ln(RV/VS)$			IR
	Mean	Std	t	Mean	Std	t	
SPX	-2.74	3.63	-8.39	-0.66	0.57	-11.83	0.98
OEX	-2.36	3.57	-7.02	-0.58	0.56	-10.34	0.85
DJX	-2.58	3.86	-6.37	-0.61	0.58	-9.07	0.87
NDX	-2.43	10.24	-2.54	-0.28	0.47	-6.49	0.55
MSFT	-3.20	12.31	-3.32	-0.30	0.52	-6.62	0.55
INTC	2.49	19.07	1.34	-0.02	0.51	-0.44	0.04
IBM	-1.68	10.24	-1.80	-0.24	0.60	-4.35	0.36
AMER	-3.51	23.76	-2.05	-0.17	0.57	-3.79	0.33
DELL	-4.43	21.35	-2.15	-0.23	0.55	-4.17	0.36
CSCO	-2.30	20.31	-1.42	-0.27	0.83	-4.06	0.36
GE	-2.24	7.63	-3.52	-0.25	0.49	-5.60	0.51
...							

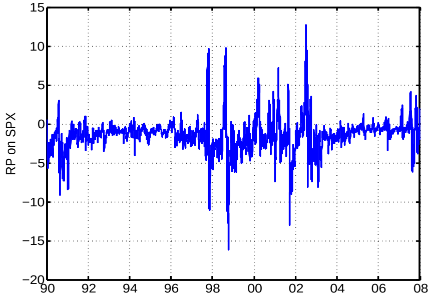
- Shorting variance swap on equity indexes make money on average.
- Variance risk premiums on single names show large cross-sectional variations — more work is needed on single names.

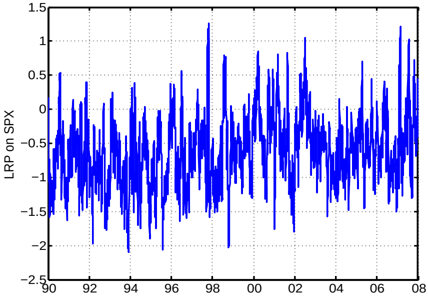
- EH1: Constant variance risk premium RP — rejected.

$$RV_{t,T} = \underset{(1.42)}{0.010} + \underset{(-4.60)}{0.455} \; VS_{t,T} + e, \quad R^2 = 26.2\%,$$

- EH2: Constant log variance risk premium LRP — not rejected.

$$\ln RV_{t,T} = \underset{(-2.59)}{-0.891} + \underset{(-0.68)}{0.919} \; \ln VS_{t,T} + e, \quad R^2 = 37.8\%,$$



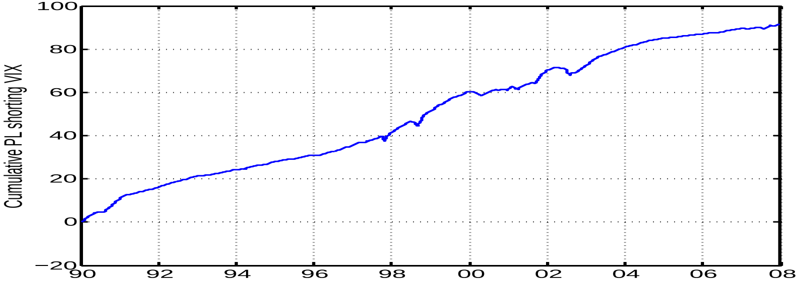


Caveat: Transaction cost will eat into the return more during low vol periods.

PL from shorting SPX variance swaps

- The negative variance risk premium on stock indexes suggests that shorting variance swap generates positive average returns.
- CBOE's VIX approximates the log profile of SPX, which we can use as an approximate quote for 30-day variance swap rate.
- Each day, short \$1 million notional of 30-day variance swap on SPX and holding the short position to maturity.

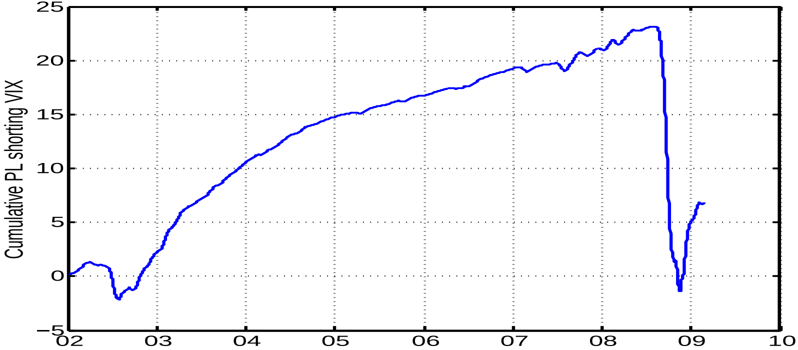
The PL from 1990 to 2007: Mean=\$1.39k, Std=\$2.17k/contract. IR= 2.2



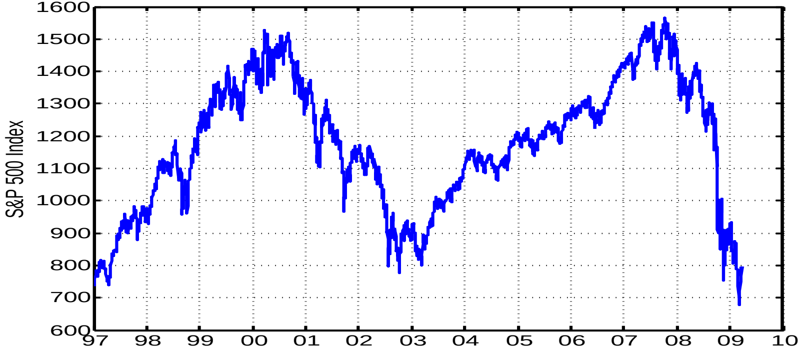
Comparison: IR from SPX: 0.4

PL from shorting SPX variance swaps

until Oct. 2008, when short-variance funds are wiped out due to leverage



But investing in SPX is even worse



Investing in both SPX and its variance swaps

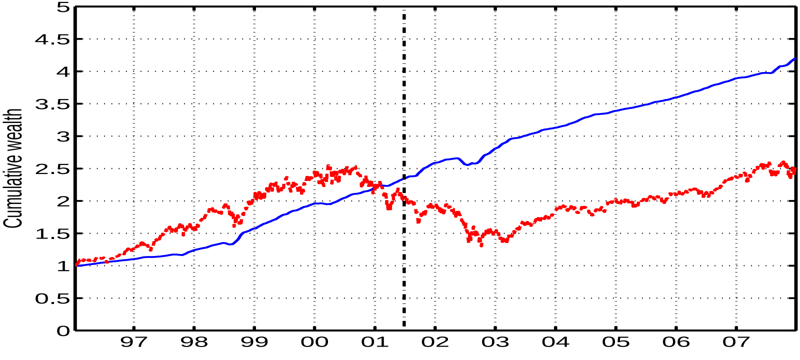
"*Optimal variance swap investments*," JFQA, with Markus Leippold.

- Analyzing variance swap rates across different maturities (2m to 24m): $\Rightarrow$ Two stochastic volatility factors: one controlling the short-term variance (v_t), the other controlling the long-term tendency (m_t).
- Consider a CRRA investor who puts a fraction of her wealth (w) in the equity index and fractions of her wealth (n_1, n_2) as notional in two variance swap contracts to span the two variance risk factors and to maximize her terminal wealth.
- Under affine structures and proportional risk premiums, the allocation weights (w, n_1, n_2) are constant over time:

$$
\begin{aligned}
w_t &= \frac{1}{\eta}\left(\gamma^S - \frac{\rho}{\sqrt{1-\rho^2}}\gamma^z\right), \\
n_{1t} &= \frac{1}{\eta D}\left[\left(\frac{\gamma^z}{\sigma_v\sqrt{1-\rho^2}} + h_v(u)\right)\phi_m(T_2-t) - \left(\frac{\gamma^m}{\sigma_m} + h_m(u)\right)\phi_v(T_2-t)\right], \\
n_{2t} &= \frac{1}{\eta D}\left[-\left(\frac{\gamma^z}{\sigma_v\sqrt{1-\rho^2}} + h_v(u)\right)\phi_m(T_1-t) + \left(\frac{\gamma^m}{\sigma_m} + h_m(u)\right)\phi_v(T_1-t)\right].
\end{aligned}
$$

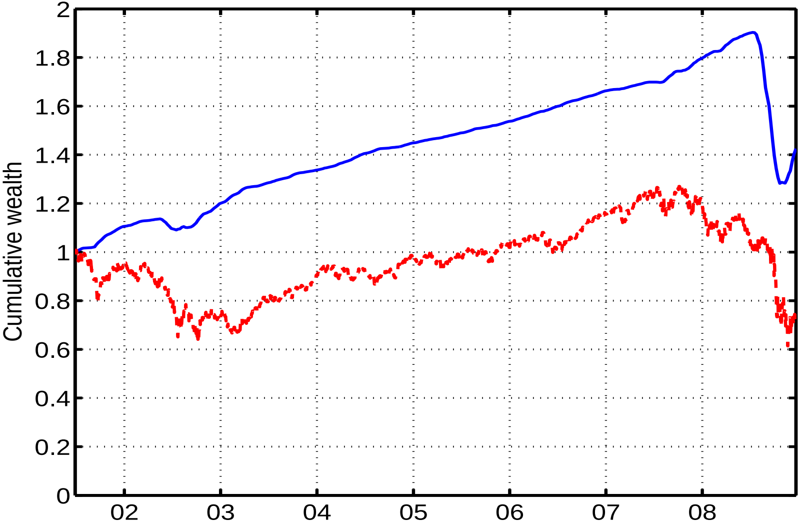
In-sample investment analysis

- Estimating the dynamics/risk premiums using data from 1996 to 2001.
  - With a relative risk aversion of 200, we have $w = -0.121$, $n_1 = -0.873$ for 2m VS, $n_2 = 0.195$ for 2y VS.
  - The high risk aversion is to counter to the extremely high variance risk premium estimate so that we can keep leverage in check.



Out-of-sample IR =2.3 vs IR for SPX alone =0.54.

Out-of-sample investment analysis



Out-of-sample IR $= 0.33$ vs IR for SPX alone $= -0.08$.

Summary: What we have learned

- Comparing variance swap rate with ex post realized variance presents a simple, direct way of measuring variance risk premium.
- The log profile matches the variance swap quote fairly well for SPX.
- Variance risk on equity index is highly priced.
  - Shorting variance swaps generates positive PL on average.
- The magnitude of the variance risk premium is proportional to the variance level.
- A portfolio that is short short-term index variance swap, long long-term variance swap, and short the equity index performs better than long the equity index alone,
  - even during the current financial meltdown.

What we still do not know

- What are the sources of the equity index variance risk premium?
  - Jump in return (even when variance is constant).
  - Return risk premium and return-variance correlation.
  - Risk premiums on independent variance risk.

  What's the relative contribution of each component?

- How to measure variance risk premiums (and their sources) on single names?
  - The possibility of default makes variance on log returns undefined.
  - Truncated payoffs in practical contracts.
  - Log profile can differ significantly from variance swap rates.
  - Default risk (premium) plays a big role.

- How to relate single name variance risk to market variance risk (in a tractable and yet reasonable way)?

- What's the guideline for variance investments across names?