Chapter 6
Efficient Diversification

Diversification and Portfolio Risk

- Don’t put all your eggs in one basket
- Effect of portfolio diversification

![Risk vs. Number of Securities](image_url)

- Diversifiable risk, non-systematic risk, firm-specific risk, idiosyncratic risk
- Non-diversifiable risk, systematic risk, market risk, covariance risk
Diversification and Portfolio Risk

- **Covariance and correlation**
  - Measure degrees of co-movement between two stocks
  - Covariance: non-standardized measure
    \[
    \text{Cov}[r_1, r_2] = E[(r_1 - \mu_1)(r_2 - \mu_2)] = E[r_1 r_2] - \mu_1 \mu_2
    \]
  - Correlation coefficient: standardized measure
    \[
    \rho_{12} = \frac{\text{Cov}[r_1, r_2]}{\sigma_1 \sigma_2} \Rightarrow \text{Cov}[r_1, r_2] = \rho_{12} \sigma_1 \sigma_2 \quad \text{and} \quad -1 \leq \rho_{12} \leq 1
    \]

Example: Covariance and Correlation

- **Calculating covariance**
  \[
  \text{Cov} [r_1, r_2] = E[(r_1 - \mu_1)(r_2 - \mu_2)] = \sum_s p(s)[(r_1(s) - \mu_1)(r_2(s) - \mu_2)]
  \]
  \[
  \text{Cov} [r_1, r_2] = E[r_1 r_2] - \mu_1 \mu_2 = \sum_s p(s)[r_1(s)r_2(s)] - \mu_1 \mu_2
  \]

<table>
<thead>
<tr>
<th>s</th>
<th>p</th>
<th>r1</th>
<th>r2</th>
<th>p*r1</th>
<th>p*r2</th>
<th>p*[r1-mu1]*2</th>
<th>p*[r2-mu2]*2</th>
<th>p*[r1<em>r2-mu1</em>mu2]</th>
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<td>-3.30</td>
<td>4.80</td>
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<td>86.70</td>
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<table>
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<th>r2</th>
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<th>p*r2</th>
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<th>p*[r1<em>r2-mu1</em>mu2]</th>
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<td>60.00</td>
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Diversification and Portfolio Risk

- A portfolio of two risky assets
  
  portfolio return: \( r_p = w_1 r_1 + w_2 r_2 \) with \( w_1 + w_2 = 1 \)
  
  - \( w_1 \): % invested in risky bond fund
  - \( w_2 \): % invested in risky stock fund

- Portfolio expected return
  
  \[ \mu_p = E[r_p] = w_1 E[r_1] + w_2 E[r_2] = w_1 \mu_1 + w_2 \mu_2 \]

- Portfolio variance
  
  \[ \sigma_p^2 = Var[r_p] = E[(r_p - \mu_p)(r_p - \mu_p)] \]
  
  \[ = w_1^2 \sigma_1^2 + w_2^2 \sigma_2^2 + 2w_1w_2 \rho_{12} \sigma_1 \sigma_2 \]

More on Portfolio Variance

- What if one of the two security is risk-free?

- What happens if two risky securities are perfectly correlated with
  
  - \( \rho_{12} = 1 \)
  - No risk reduction

- What happens if two risky securities are perfectly correlated with
  
  - \( \rho_{12} = -1 \)
Diversification and Portfolio Risk

- Example: Portfolio of two risk securities
  - $w$ in security 1, $(1-w)$ in security 2
    - $\mu_1 = 0.10$, $\sigma_1 = 0.15$
    - $\mu_2 = 0.14$, $\sigma_2 = 0.20$, $\rho_{12} = 0.2$

- Expected return (Mean):
  - $\mu_p = 0.10 \times w + 0.14 \times (1-w) = 0.14 - 0.04 \times w$

- Variance
  - $\sigma_p^2 = 0.15^2 w^2 + 0.20^2 (1-w)^2 + 2 \times 0.2 \times 0.15 \times 0.20 \times (1-w)$

- What happens when $w$ changes?
  - Expected return ↑ as weight in 1 ↓
  - What about variance … first ↓, then ↑

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Mean-variance Frontier

- $w$ moves from 1 → 0

<table>
<thead>
<tr>
<th>$w$</th>
<th>1-w</th>
<th>$E(rp)$</th>
<th>$Var(rp)$</th>
<th>std dev</th>
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GMVP: Global Minimum Variance Portfolio
Efficient Portfolio Frontier

- What’s special about a portfolio with 67% in Security 1 and 33% in Security 2?
- Efficient portfolio has < 67% in 1, and > 33% in 2

The effect of correlation
- Lower correlation means greater risk reduction
- If $\rho = +1.0$, no risk reduction is possible
Efficient Portfolio Frontier

- Efficient portfolio of many securities
  - $E[r_p]$ is the weighted average of $N$ securities
  - $\sigma_p^2$ is the sum of all weighted pair-wise covariance measures
    - Covariance with a security itself is variance
- Optimal combination leads to
  - Lowest risk for a given level of expected return
  - Highest expected return at a given risk level
- Efficient frontier describes the optimal risk-return trade-off
  - Portfolios not on efficient frontier are dominated

Efficient Frontier

- Efficient frontier
- Global minimum variance portfolio
- Minimum variance frontier
- Individual assets
- St. Dev.

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Efficient Frontier with A Riskfree Asset

- With risky assets only
  - No portfolio with zero variance (if no two securities are perfectly correlated)
  - GMVP has the lowest variance
- With an additional riskfree asset
  - Can reduce risk further without sacrificing expected return
    - Zero variance if investing in riskfree asset only
    - How will the efficient frontier change?
Efficient Frontier with A Riskfree Asset

- CAL(O) dominates other lines
  - Best risk and return trade-off
  - Steepest slop (highest Sharpe ratio)

\[ S_p = \frac{E[r_p] - r_f}{\sigma_p} > \frac{E[r_s] - r_f}{\sigma_s} \]

- Portfolios along CAL(O) has the same Sharpe ratio
- No portfolio with higher Sharpe ratio is achievable
- Dominance independent of risk preference

Optimal CAL

- What’s so special about portfolio (O)?
  - The optimal portfolio is the market portfolio
  - Mutual fund theorem: An index mutual fund (market portfolio) and T-bills are sufficient for investors
  - Investors adjust the holding of index fund and T-bills according to their risk preferences
    - Where CML meets the indifference curve
Portfolio Selection and Risk Aversion

\[ E[R_p] \quad \sigma_p \]

Indifference curve

CML

Mean-Variance Frontier

Two Step Portfolio Allocation

- **Step 1:** Determine the optimal risky portfolio
  - Get the optimal mix of risky stocks and bonds.
  - Optimal for all investors regardless of risk aversion

- **Step 2:** Determine the best complete portfolio
  - Obtain the best mix of the optimal risky portfolio and T-Bills.
  - Different investors may have different best complete portfolios
  - Depend on risk aversion.
Wrap-up

- How to calculate portfolio return and risk?
- How to calculate covariance and correlation coefficient based on scenario analysis
- How to calculate covariance based on portfolio weights
- What is the mean-variance frontier with risky assets with different correlations between risky assets?
- What is the efficient portfolio frontier
  - with risky assets
  - With both risky assets and riskfree asset
- Why do portfolios on the efficient frontier dominate other possible combinations?