Measuring financial risk

Sebastiano Manzan

ECO 4051 | Spring 2017
Why measure and manage risk?

- The risk-return tradeoff is at the core of any business and financial decision.
- Modern companies and financial institutions are large enterprises with many (thousands?) of individuals making decisions in an uncertain environment.
- How can we measure, aggregate, and manage (= control) the overall risk exposure of an organization?
- What type of risks are we trying to measure and control?
Types of financial risk

- **Market risk**: losses due to movements in the level and/or volatility of asset prices
  - *directional*: exposure to movements of the market
  - *non-directional*: exposure to changes of volatility

- **Credit risk**: losses arising from defaults on bonds and loans; changes to credit ratings or market perception of default (sub-prime, mortgage-backed securities etc)

- **Liquidity risk**:
  - *Asset liquidity risk*: arises when transactions cannot be executed at prevailing market prices due to size of the position or thinness of the market
  - *Funding liquidity risk*: inability to meet margin calls on leveraged portfolios (LTCM)

- **Operational Risk**: due to human or technical mistakes, employees’ fraud, inadequacy of risk management procedures, inadequacy of pricing models (model risk)
In the last 20 years there have been many cases in which failure to measure and manage risk lead to huge losses or even bank failures:

- Leeson/Barings: loss of $1.3 billion and that wiped out the capital of Barings; shareholders lost everything, bondholders almost everything

- Orange County: loss of $1.6 billion

- LTCM: loss of $4 billion


- 2008 Financial Crisis: $4 trillion?
Leeson/Barings (1994-1995)

- Large long bet on the Japanese equity market through NIKKEI 225 index futures (notional $7 billion) and sold options betting on a stable market.

- At the beginning of 1995 the Japanese equity market lost more than 15% and Leeson’s position started bleeding.

- The total loss was of $1.3 billion that wiped out the equity capital of the bank which was sold to ING for £1.

- Leeson was able to accumulate such a large directional bet for lack of internal controls.

- Operational and market risks played a role in Barings disaster.
Many of the financial disasters are due to the use of derivatives and leverage. Derivatives are used to hedge but also speculate (directional bets) while leverage is used to boost profits from limited capital. The problem is not so much the use of derivatives and/or leverage, rather the *uncontrolled* risk-taking allowed by these positions. Good risk management practice means *measuring and managing* these risk (not *eliminating* it!)
The increasingly frequent financial disasters has called the attention of the profession and regulators on the need for a more stringent risk management practice.

Professional initiatives:

- Derivatives Policy Group
  - management controls
  - enhanced reporting
  - evaluation of risk versus capital
  - measure market risk via Value at Risk (VaR) at 99% level and 2-week horizon

- RiskMetrics from J.P. Morgan
  - established a framework for VaR calculation

Regulators:

- Basel Accord on Banking Supervision (1988 and later amendments)
The Accords set rules on bank **capital adequacy** as a way to monitor/control the amount of risk-taking.

Banks are allowed to use their internal models to determine the capital charge.

They should demonstrate to have an appropriate risk management system, conduct regular stress testing, having independent risk-control unit and external audits.

Each bank is subject to the **Market Risk Charge (MRC)** which is calculated as:

\[
MRC = k \times \max \left[ \text{yesterday’s VaR}, \text{60-days average VaR} \right]
\]

where **VaR** is defined as **maximum loss that we expect at a certain confidence level and target horizon**.
The calculation of MRC is based on the following parameters:

- **Horizon**: 10 trading days (2 weeks); allowed to scale up the 1-day VaR using $\sqrt{10}$
- **Confidence level**: 99%
- **Estimation**: at least 1 year of data and update of estimates at least once a quarter
- **Correlation**: can be accounted among and across broad categories (bonds, equities, commodities, etc.)
- **Multiplicative Factor $k$**: at least 3
- **Plus Factor**: $k$ can be increased by regulators to values higher than 3 if backtesting indicates inadequacy of the internal risk model
What is Value-at-Risk (VaR)?

▶ Denote the profit/loss of a financial institution in day \( t + 1 \) by 
\[ R_{t+1} = 100 \times \ln\left(\frac{W_{t+1}}{W_t}\right), \]
where \( W_{t+1} \) is the portfolio value in day \( t + 1 \).

▶ We define **Value-at-Risk (VaR)** as the *maximum loss that it is expect at a certain confidence level and target horizon*; using a 99% confidence level and 1-day horizon VaR for day \( t + 1 \), \( \text{VaR}_{t+1} \), is defined as

\[
P(R_{t+1} > \text{VaR}_{t+1}^{0.99}) = 0.99
\]

▶ \( \text{VaR} \) can also be expressed as the minimum loss that we expect at a certain confidence level and target horizon, that is,

\[
P(R_{t+1} \leq \text{VaR}_{t+1}^{0.99}) = 0.01
\]

In statistical terms, \( \text{VaR} \) is the 0.01 quantile of the probability distribution of \( R_{t+1} \).

▶ For a portfolio value of \( W_t \) the $\text{VaR}$ is calculated as

\[
\text{VaR}_{t+1}^{0.99} = W_t \times (\exp(|\text{VaR}_{t+1}^{0.99}|/100) - 1)
\]
- **VaR** at $100 \times (1 - \alpha)\%$ represents the $100 \times \alpha\%$ quantile of the profit/loss distribution.
VaR assuming normality

- Assume that $R_{t+1} \sim N(\mu, \sigma^2)$ with constant mean and variance

- Based on this assumption the $P(R_{t+1} \leq \text{VaR}_{1-\alpha}) = \alpha$, with $1 - \alpha = 0.99$ is

$$\text{VaR}_{t+1}^{0.99} = \mu - 2.33 \times \sigma$$

and, more generally,

$$\text{VaR}_{t+1}^{1-\alpha} = \mu + z_\alpha \times \sigma$$

where $z_\alpha$ represents the quantile from the standard normal distribution at level $\alpha$

- Example: we hold a portfolio that replicates the S&P 500 Index and we obtain historical data for the Index from Jan 03, 1990, to May 08, 2017. R implementation:

```r
mu = mean(sp500daily)
sigma = sd(sp500daily)
var = mu + qnorm(0.01) * sigma  # qnorm(0.01) = -2.33
```

[1] -2.569
The assumption that $R_{t+1} \sim N(\mu, \sigma^2)$ restricts the mean and the variance of the portfolio return to be constant over time.

This is an unrealistic assumption given the evidence that daily returns show significant variation of volatility over time.

We can thus assume that $R_{t+1} \sim N(\mu_{t+1}, \sigma_{t+1}^2)$ where:

- **Conditional mean** $\mu_{t+1}$: an AR(1) process, i.e., $\mu_{t+1} = \phi_0 + \phi_1 R_t$
- **Conditional variance** $\sigma_{t+1}$: MA, EMA, GARCH(1,1), or GJR-GARCH(1,1)

Value-at-Risk is then calculated as: $\text{VaR}_{t+1}^{0.99} = \mu_{t+1} - 2.33\sigma_{t+1}$.
Model: \( R_{t+1} = \mu + \sigma_{t+1} \epsilon_{t+1} \)

- \( \sigma_{t+1} \) estimated by EMA(0.06)
- \( \epsilon_{t+1} \sim N(0,1) \)

For 1.074% of the 6891 days the return was smaller relative to VaR

```r
library(TTR)
mu <- mean(sp500daily)
sigmaEMA <- EMA((sp500daily - mu)^2, ratio=0.06)^0.5
var <- mu + qnorm(0.01) * sigmaEMA
```
Expected Shortfall (ES)

- VaR represents the **maximum (minimum) loss that is expected with 99% (1%) probability**

- However, it does not provide a measure of how large the losses are likely to be in case an extreme event happens

- An alternative risk measure is **Expected Shortfall (ES)** that is defined as

\[
ES_{t+1}^{1-\alpha} = E(R_{t+1}|R_{t+1} \leq VaR_{t+1}^{1-\alpha})
\]

and represents the expected return conditional on the return being smaller than \(100\%(1-\alpha)\%\) VaR
If we assume that returns are normally distributed, $ES$ is given by

$$ES_{t+1}^{1-\alpha} = -\sigma_{t+1} \frac{\phi \left(VaR_{t+1}^{1-\alpha}/\sigma_{t+1}\right)}{\alpha} = -\sigma_{t+1} \frac{\phi \left(z_\alpha\right)}{\alpha}$$

where $\phi(\cdot)$ represents the PDF of the standard normal distribution.

If we are using $1 - \alpha = 0.99$ then we have that (for $\mu = 0$):

- $VaR_{t+1}^{0.99} = -2.33\sigma_{t+1}$
- $\Phi^{-1}(VaR_{t+1}^{0.99}/\sigma_{t+1}) = \phi(-2.33) = 0.026$
- $ES_{t+1}^{0.99} = -\sigma_{t+1} \frac{\phi(-2.33)}{0.01} = -2.643\sigma_{t+1}$

Given the same level $\alpha$, $ES$ provides a more conservative risk measure relative to $VaR$. 

\[ \]
Density and Distribution functions for the standard normal distribution \( x \sim N(0, 1) \)

- \( \Phi(-2.33) = \)
- \( \Phi(2.33) = \)
- \( \Phi^{-1}(0.01) = \)
- \( \Phi^{-1}(0.99) = \)
\( \text{sigma} = 1 \)
\( \text{alpha} = \text{seq}(0.001, 0.05, \text{by}=0.001) \)
\( \text{ES} = - \frac{\text{dnorm}(\text{qnorm}(\text{alpha}))}{\alpha} \times \text{sigma} \)
\( \text{VaR} = \text{qnorm}(\alpha) \times \text{sigma} \)
So far we discussed how to calculate VaR at the 1-day horizon but regulators ask financial institutions to report their potential losses over a 10 day horizon.

Basel rules allow banks to scale-up the one-day risk measures by $\sqrt{10}$.

Where does this $\sqrt{10}$ rule come from?

The return of holding the portfolio in the following $K$ days is

$$R_{t+1:t+k} = \sum_{k=1}^{K} R_{t+k} = R_{t+1} + \cdots + R_{t+K}$$
Assuming that the daily returns are *independent* and *identically distributed* (i.i.d.) with mean $\mu$ and variance $\sigma^2$, then we have that the expected value of the multi-period return is

$$E \left( \sum_{k=1}^{K} R_{t+k} \right) = \sum_{k=1}^{K} \mu = K \mu$$

and its variance is

$$Var \left( \sum_{k=1}^{K} R_{t+k} \right) = \sum_{k=1}^{K} \sigma^2 = K \sigma^2$$

So that the standard deviation of the $R_{t+1:t+k}$ is equal to $\sqrt{K} \sigma$

Expected return for unit of risk:

$$E(R_{t+1:t+K})/\sqrt{Var(R_{t+1:t+K})} = \frac{K\mu}{\sqrt{K} \sigma} = \sqrt{K} \frac{\mu}{\sigma}$$

$VaR$ at 1% of the multi-period return $R_{t+1:t+K}$ is given by

$$VaR_{t+1:t+K}^{1-\alpha} = K \mu - 2.33 \sqrt{K} \sigma$$
The $\sqrt{10}$ rule relies on the assumption that daily returns are independent over time.

What happens if this assumption fails to hold?

Assume that $K = 2$ so that $R_{t+1:t+2} = R_{t+1} + R_{t+2}$.

The variance of the multiperiod return $Var(R_{t+1} + R_{t+2})$ is equal to

$$Var(R_{t+1}) + Var(R_{t+2}) + 2Cov(R_{t+1}, R_{t+2})$$

$$= \sigma^2 + \sigma^2 + 2\rho\sigma^2 = 2\sigma^2(1 + \rho)$$

If $\rho > 0$ the variance (and thus $VaR$) of the multiperiod return is larger relative to the case with no correlation since $2\sigma^2(1 + \rho) > 2\sigma^2$.

Bottom line: assuming independence we under-estimate the risk in our portfolio if returns are actually positively correlated.
VaR assuming non-normality

- *Fat tails* is the characteristic of daily financial returns to have events 3 (or larger) standard deviations away from the mean more often relative to the normal distribution.

- Is time-varying volatility a possible explanation for the *non-normality* in the data?

- This could be the case if returns are switching between two regimes of high and low volatility.

- The sample skewness of the *standardized residuals* is \(-0.431\) and the sample excess kurtosis is \(2.261\).

- In this case, there is still some skewness and, more generally, there might still be some non-normality in the standardized residuals.

- We can depart from the assumption of normality in two ways:
  - *Cornish-Fisher* approximation
  - *t* distribution
The **Cornish-Fisher** (CF) approximation consists of a Taylor expansion of
the normal density which has the effect of making the approximate density a
function of skewness and kurtosis

\[ \text{VaR} = \frac{1}{\sigma_t + 1} \] (CF

where \( \sigma_t \) is the standard deviation at time \( t \) and \( \alpha \) is the confidence level.

\( z_{\alpha}^{CF} = z_{\alpha} + \frac{SK}{6} (z_{\alpha}^2 - 1) + \frac{EK}{24} (z_{\alpha}^3 - 3z_{\alpha}) + \frac{SK^2}{36} (2(z_{\alpha})^5 - 5z_{\alpha}) \)

where \( SK \) and \( EK \) represent the skewness and excess kurtosis, respectively, and \( z_{\alpha} \)
represents the \( \alpha \) quantile

- If \( SK = EK = 0 \) then \( z_{\alpha}^{CF} = z_{\alpha} = -2.33 \) (normal)

- For \( \alpha = 0.01 \) we have that \( z_{\alpha} = -2.33 \) and \( z_{\alpha}^{CF} \) is then equal to:

\[ z_{\alpha}^{CF} = -2.33 + 0.7 \times SK - 0.23 \times EK - 3.46 \times SK^2 \]
Effect of changing skewness and kurtosis on $z_{CF}$
A distribution with fat tails

- An alternative approach is to discard the normal distribution in favor of a distribution with fat tails such as the $t$ distribution with $d$ degrees-of-freedom.

- If we assume that $\epsilon_{t+1} \sim t_d$ in $R_{t+1} = \sigma_{t+1}\epsilon_{t+1}$ we can estimate the volatility parameters and $d$ by ML.

```r
q1 <- ggplot(data.frame(x = c(-8, 8)), aes(x = x)) +
    stat_function(fun = dnorm, aes(colour = "N(0,1)") ) +
    stat_function(fun = dt, args = list(df=4), aes(colour = "t(4)") ) +
    stat_function(fun = dt, args = list(df=10), aes(colour = "t(10)") ) +
    theme_bw() + labs(x = "Probability", y = NULL) + coord_cartesian(x=c(-3,3)) +
    scale_colour_manual("Distribution", values = c("seagreen3", "steelblue3", "tomato3"))
q2 <- q1 + coord_cartesian(x=c(-8, -3), y=c(0,0.03))
```
A simple way to obtain a rough estimate of the degrees of freedom is 
\[ d = \frac{6}{E K} + 4, \] where \( E K \) is the excess kurtosis of the returns.

This rule is based on the fact that the \( t_d \) distribution (for \( d > 4 \)) has excess kurtosis equal to \( \frac{6}{(d - 4)} \) (and it is infinite for \( 2 < d < 4 \)).

The standardized returns have sample excess kurtosis of 2.26 so that the estimate of \( d \) is equal to 6.65 which can be rounded to 7.

This value confirms the evidence that the returns standardized by the volatility forecast still deviate from normality, in particular on the left tail.
Historical Simulation (HS)

- The risk models that we discussed are based on two building blocks:
  - mean and volatility models (constant or AR(p); EMA, GARCH(1,1), or GJR-GARCH)
  - error distribution (standard normal, $t$)

- Historical Simulation (HS) is an alternative approach that calculates VaR at $\alpha$ level as the $1 - \alpha$ quantile of the most recent $M$ days

- HS does not make any assumptions about the mean/volatility model and/or the error distribution

- In this sense, it can be considered a non-parametric approach because of the lack of parametric assumptions

- A typical value for $M$ is 250 days (one trading year)
M1 = 250  
M2 = 1000  
alpha = 0.01  

hs1 <- rollapply(sp500daily, M1, quantile, probs=alpha, align="right")  
hs2 <- rollapply(sp500daily, M2, quantile, probs=alpha, align="right")
Zoom on the financial crisis period 2008-2009
Comparison of VaR based on EMA (with normal errors) and HS during the financial crisis

EMA is clearly faster in adapting to changes in volatility

Percentage of violations from January 03, 1990 to May 08, 2017:

- HS: 1.3061%
- EMA: 1.0013%
Pros of HS:

- No model required for the volatility dynamics and error distribution
- Easy and fast to calculate, no estimation required

Cons of HS:

- Choice of the estimation window $M$
- Transforming the 1-day $\text{VaR}$ to 10-day $\text{VaR}$ (practice: multiply by $\sqrt{10}$)
Monte Carlo (MC) Simulation

- In the 1990s, EMA and HS were the most widely used methods to calculate VaR.
- Both for parametric and non-parametric methods, we need to use approximations to derive the distribution of the cumulative return.
- This task is much simplified by considering simulation-based methods.
- Simulation methods are based on the following idea: simulate data from the estimated risk model (volatility model + error distribution) to determine the distribution of the cumulative return at horizon $K$. 
One application of MC methods is to simulate the future distribution of returns from 1 to $K$ periods ahead.

We consider two dates:

- January 22, 2007: period of low volatility
- September 29, 2008: period of high volatility
We are at the closing of day $T$ and we want to forecast the distribution of the portfolio profit/loss from 1 to $K$ periods ahead. A MC simulation is composed of the following steps:

1. Estimate the volatility model $R_{t+1} = \sigma_{t+1}\epsilon_{t+1}$ (e.g., GARCH(1,1) + $\epsilon_{t+1} \sim N(0, 1)$) and produce a forecast $\sigma_{T+1}$

2. Generate $S$ random draws for the error denoted $\epsilon_{s,T+1}$ (for $s = 1, \cdots, S$) and $S$ random returns for $T + 1$ as $R_{s,T+1} = \sigma_{T+1}\epsilon_{s,T+1}$

3. Produce a forecast $\sigma_{s,T+2}$ as $\sigma^2_{s,T+2} = \omega + \alpha R^2_{s,T+1} + \beta \sigma^2_{T+1}$

4. Generate other $S$ random draws for the error denoted $\epsilon_{s,T+2}$ and $S$ random returns for $T + 2$ as $R_{s,T+2} = \sigma_{s,T+2}\epsilon_{s,T+2}$

5. Produce a forecast $\sigma_{s,T+3}$ as $\sigma^2_{s,T+3} = \omega + \alpha R^2_{s,T+2} + \beta \sigma^2_{s,T+2}$

6. Repeat 4-5 until the forecast horizon $K$
Step 1: estimate GARCH model and produce the forecast for T+1

```r
require(rugarch)
spec = ugarchspec(variance.model=list(model="sGARCH", garchOrder=c(1,1)),
               mean.model=list(armaOrder=c(0,0), include.mean=FALSE))
fitgarch = ugarchfit(spec = spec, data = window(sp500daily, end=lowvol))

omega  alpha1  beta1
0.005   0.054   0.942

2007-01-22
T+1     0.483

fitgarch1 = ugarchfit(spec = spec, data = window(sp500daily, end=highvol))

omega  alpha1  beta1
0.006   0.059   0.937

2008-09-29
T+1     3.085
```
Step 2-6 Iterate the simulation step until K days ahead

```r
set.seed(9874)

S = 10000  # number of MC simulations
K = 250    # forecast horizon

# create the future dates
lowvol <- as.Date("2007-01-22")
futdates <- lowvol + 1:K

# create the matrices to store the simulated return and volatility
R = xts(matrix(sigma*rnorm(S), K, S, byrow=TRUE), order.by=futdates)
Sigma = xts(matrix(sigma, K, S), order.by=futdates)

# iteration to calculate R and Sigma based on the previous day
for (i in 2:K)
{
    Sigma[i,] = (gcoef["omega"] + gcoef["alpha1"] * R[i-1,]^2 + gcoef["beta1"] * Sigma[i-1,]^2)^0.5
    R[i,] = rnorm(S) * Sigma[i,]
}
```

- The assumption that the error distribution is \( N(0, 1) \) can be easily changed by replacing `rnorm(S)` with, e.g., `rt(S, df=6)` to simulate from \( t_6 \)
At each horizon $k$ plot the quantiles (across $S$) at 0.10, · · · , 0.90
For the simulated volatility we also plot the average (across $S$)
MC cumulative returns

- The cumulative return is defined as \( R_{t+1:t+k} = \sum_{j=1}^{k} R_{T+j} \).
- MC VaR is represented by the \( 1 - \alpha \) quantile of the simulated returns at each horizon \( k \).

```r
# Calculate cumulative returns
Rcum <- cumsum(R)
Rcumq <- xts(t(apply(Rcum, 1, quantile, probs=c(0.01, 0.05, seq(0.10, 0.90, 0.10), 0.95, 0.99))), order.by=futdates)

# Calculate cumulative returns for future dates
R1cum <- cumsum(R1)
R1cumq <- xts(t(apply(R1cum, 1, quantile, probs=c(0.01, 0.05, seq(0.10, 0.90, 0.10), 0.95, 0.99))), order.by=futdates1)
```
The simulated return values can also be used to calculate ES by averaging (at each $k$) the returns values that are smaller or equal to VaR

$$\text{VaR} = \text{xts}\left(\text{apply}(\text{Rcum}, 1, \text{quantile}, \text{probs}=0.01), \text{order.by}=\text{futdates}\right)$$

$$\text{VaRmat} = \text{matrix}(\text{VaR}, K, S, \text{byrow}=\text{FALSE})$$

$$\text{Rviol} = \text{Rcum} \times (\text{Rcum} < \text{VaRmat})$$

$$\text{ES} = \text{xts}\left(\frac{\text{rowSums}(\text{Rviol})}{\text{rowSums}(\text{Rviol} \neq 0)}, \text{order.by}=\text{futdates}\right)$$
Filtered Historical Simulation (FHS)

▶ A drawback of HS is that it adapts very slowly to changes in asset volatility

▶ An alternative that has been proposed is Filtered Historical Simulation (FHS) which combines features of parametric modeling, MC, and HS

▶ FHS assumes:

1. A time-series volatility model for returns, e.g., \( R_{t+1} = \sigma_{t+1} \epsilon_{t+1} \), where \( \sigma_{t+1} \) is assumed to follow a GARCH model
2. No error distribution is assumed; instead, the simulated errors \( \epsilon_{s,t+k} \) are random draws from the standardized returns \( R_{t+1}/\sigma_{t+1} \)
3. 1 and 2 are used in a MC simulation as shown above
The only difference between MC and FHS consists of the distribution assumed to draw the simulated errors: a parametric distribution in the case of MC (standard normal or $t_d$) or the empirical distribution of the standardized returns ($R_{t+1}/\sigma_{t+1}$).

This implies that the resampled $\epsilon_{s,t+k}$ maintain the features of the data, such as skewness and kurtosis.

In R this is implemented by drawing the $\epsilon_{s,t+k}$ by `sample(std.resid, S, replace=TRUE)` instead of `rnorm(S)` (or `rt(S, df)`).

In the next plots we compare the expected one-day volatility and the 99% VaR based on the $k$-period cumulative return conditional on January 22, 2007 and September 29, 2008.
# standardized residuals
std.resid = as.numeric(residuals(fitgarch, standardize=TRUE))
std.resid1 = as.numeric(residuals(fitgarch1, standardize=TRUE))

for (i in 2:K)
{
    Sigma.fhs[i,] = (gcoef['omega'] + gcoef['alpha1'] * R.fhs[i-1,]^2 +
                    gcoef['beta1'] * Sigma.fhs[i-1,]^2)^0.5
    R.fhs[i,] = sample(std.resid, S, replace=TRUE) * Sigma.fhs[i,]
    Sigma.fhs1[i,] = (gcoef1['omega'] + gcoef1['alpha1'] * R.fhs1[i-1,]^2 +
                      gcoef1['beta1'] * Sigma.fhs1[i-1,]^2)^0.5
    R.fhs1[i,] = sample(std.resid1, S, replace=TRUE) * Sigma.fhs1[i,]
}
In practice, the portfolio of large financial institutions is composed of several assets and its return can be express as

\[ R_{p,t} = \sum_{j=1}^{J} w_{j,t} R_{j,t} \]

where:
- \( R_{p,t} \) represents the portfolio return in day \( t \)
- \( w_{j,t} \) is the weight of asset \( j \) in day \( t \) (and there is a total of \( J \) assets)
- \( R_{j,t} \) is the return of asset \( j \) in day \( t \)

Let’s assume that the bank holds only 2 assets (i.e., \( J = 2 \))

The expected portfolio return is given by: \( E(R_{p,t}) = \mu_{p,t} = w_{1,t}\mu_1 + w_{2,t}\mu_2 \)

The portfolio variance is given by:
\[
Var(R_{p,t}) = \sigma_{p,t}^2 = w_{1,t}^2 \sigma_1^2 + w_{2,t}^2 \sigma_2^2 + 2 w_{1,t} w_{2,t} \rho_{12} \sigma_1 \sigma_2
\]
which is a function of the individual variances and the correlation between the two assets, \( \rho_{12} \).
The portfolio Value-at-Risk is then given by

\[
VaR_{p,t}^{1-\alpha} = \mu_{p,t} + z_\alpha \sigma_{p,t}
\]

\[
= w_1, t \mu_1 + w_2, t \mu_2 + z_\alpha \sqrt{w_1^2, t \sigma_1^2 + w_2^2, t \sigma_2^2 + 2w_1, t w_2, t \rho_{12} \sigma_1 \sigma_2}
\]

If we assume that \( \mu_1 = \mu_2 = 0 \), the \( VaR_{p,t}^{1-\alpha} \) formula can be expressed as follows:

\[
VaR_{p,t}^{1-\alpha} = +z_\alpha \sqrt{w_1^2, t \sigma_1^2 + w_2^2, t \sigma_2^2 + 2w_1, t w_2, t \rho_{12} \sigma_1 \sigma_2}
\]

\[
= -\sqrt{z_\alpha^2 w_1^2, t \sigma_1^2 + z_\alpha^2 w_2^2, t \sigma_2^2 + 2 \times z_\alpha^2 w_1, t w_2, t \rho_{12} \sigma_1 \sigma_2}
\]

\[
= -\sqrt{(VaR_{1,t}^{1-\alpha})^2 + (VaR_{2,t}^{1-\alpha})^2 + 2 \times \rho_{12} \times VaR_{1,t}^{1-\alpha} \times VaR_{2,t}^{1-\alpha}}
\]

and:

1. if \( \rho_{12} = 1 \): \( VaR_{1,t}^{1-\alpha} + VaR_{2,t}^{1-\alpha} \)
2. if \( \rho_{12} = -1 \): \( -|VaR_{1,t}^{1-\alpha} - VaR_{2,t}^{1-\alpha}| \)
3. if \( -1 < \rho_{12} < 1 \):
   \( VaR_{1,t}^{1-\alpha} + VaR_{2,t}^{1-\alpha} < VaR_{p,t}^{1-\alpha} < -|VaR_{1,t}^{1-\alpha} - VaR_{2,t}^{1-\alpha}| \)
If we assume that the mean and the variance vary over time then $VaR_{p,t}^{1-\alpha}$ is:

$$VaR_{p,t}^{1-\alpha} = w_1,t \mu_{1,t} + w_2,t \mu_{2,t} + z_\alpha \sqrt{w_1^2,t \sigma_{1,t}^2 + w_2^2,t \sigma_{2,t}^2 + 2w_1,t w_2,t \rho_{12,t} \sigma_{1,t} \sigma_{2,t}}$$

where:

- $\mu_{1,t} = \beta_0 + \beta_1 R_{1,t-1}$ and $\mu_{2,t} = \beta_0 + \beta_1 R_{2,t-1}$
- $\sigma_{1,t}^2, \sigma_{2,t}^2$ MA, EMA, and a GARCH
- $\rho_{12,t} ??$
Modeling correlations

- A simple approach to modeling correlations consists of using MA and EMA smoothing as in the case of modeling volatility.

- Denote the returns of asset 1 by $R_{1,t}$ and of asset 2 by $R_{2,t}$.

- The $MA(M)$ estimate of the covariance of the two assets is:

$$\sigma_{12,t+1} = \frac{1}{M} \sum_{m=1}^{M} R_{1,t-m+1} R_{2,t-m+1}$$

- The correlation is then given by: $\rho_{12,t+1} = \sigma_{12,t+1} / (\sigma_{1,t+1} \cdot \sigma_{2,t+1})$

- If the portfolio is composed of $J$ assets there are $J \cdot (J - 1) / 2$ correlations to estimate.

- An alternative approach is to use EMA smoothing which can be implemented using the recursive formula discussed earlier:

$$\sigma_{12,t+1} = (1 - \lambda) \sigma_{12,t} + \lambda R_{1,t} R_{2,t}$$

and the correlation can be obtained as earlier by dividing with the standard deviation forecasts.
We hold a portfolio that is invested for a fraction $w_1$ in a gold ETF (ticker: GLD) and the remaining fraction $1 - w_1$ in the S&P 500 ETF (ticker: SPY).

The data are from January 02, 1970, January 02, 1970 until January 03, 1970, January 02, 1970 at the daily frequency.

I will forecast volatility and correlation using the EMA approach with $\lambda = 0.94$.

data <- getSymbols(c("GLD", "SPY"), from="2005-01-01")
R <- 100 * merge(Cl(Cl(GLD)), Cl(Cl(SPY)))
names(R) <- c("GLD", "SPY")

# EMA for the product of returns
prod <- R[,1] * R[,2]
cov <- EMA(prod, ratio=0.06)

# EMA for the squared returns
sigma <- do.call(merge, lapply(R^2, FUN = function(x) EMA(x, ratio=0.06)))^0.5
names(sigma) <- names(R)

corr <- cov / (sigma[,1] * sigma[,2])
autoplot(corr) + geom_hline(yintercept=0, color="tomato3", linetype="dashed") + geom_hline(yintercept = as.numeric(cor(R[,1], R[,2], use="pairwise.complete"), color="seagreen4", linetype="dashed") + theme_bw() + labs(x=NULL, y=NULL)
VAR for a portfolio investing $w_1$ in GLD and $1 - w_1$ in SPY

\[
\begin{align*}
  w_1 &= 0.5 \quad \# \text{weight of asset 1} \\
  w_2 &= 1 - w_1 \quad \# \text{weight of asset 2} \\
  \text{VAR} &= -2.33 \times \left( (w_1 \times \text{sigma}[1])^2 + (w_2 \times \text{sigma}[2])^2 + 2 \times w_1 \times w_2 \times \text{corr} \times \text{sigma}[1] \times \text{sigma}[2] \right)^{0.5} \\
  \text{autoplot(VAR)} + \text{geom_hline(yintercept=0, color="tomato3", linetype="dashed")} + \\
  \text{theme_bw()} + \text{labs(x=NULL, y=NULL)}
\end{align*}
\]
A comparison of portfolio \textit{VaR} with \textit{VaR} for a portfolio that is fully invested in GLD or SPY

\texttt{VaRGLD} = -2.33 \times \text{sigma}[1]
\texttt{VaRSPY} = -2.33 \times \text{sigma}[2]
\texttt{mydata} <- \texttt{merge(VaR, VaRSPY, VaRGLD)}
\texttt{autoplot(mydata, facets=FALSE) + theme_bw() + geom_hline(yintercept=0, color="tomato2", linetype="dashed")}
What makes a risk model a good model?

We need to identify some characteristics which should hold if our model is the true one and test them on the data.

Backtesting means testing the goodness of our risk model in terms of three properties of the violations process, $V_{t+1}$, defined as

$$V_{t+1}^{1-\alpha} = I(R_{t+1} \leq VaR_{t+1}^{1-\alpha})$$

where $I(A)$ takes value 1 if the statement $A$ is true, and 0 otherwise.

These properties are:

- **Unconditional Coverage**: we expect that $100 \times (1 - \alpha)\%$ VaR is violated $100\times\alpha\%$ of the days
- **Independence**: violations should be independent over time
- **Conditional Coverage**: independent of what happened the previous day, we should always have a 1% chance of having a violation
For the S&P 500 example above with volatility predicted using EMA and \( \alpha = 0.01 \) we have:

\[
\begin{align*}
T_1 &= \text{sum}(V) \\
TT &= \text{length}(V) \\
\hat{\alpha} &= T_1 / TT
\end{align*}
\]

\[\text{round}(\hat{\alpha}, 3)\]

[1] 0.011

The fraction of days with violations is 1.079\% which is different from the expected value of \( \alpha = 1\% \)

Significantly different? We want to evaluate this hypothesis statistically by testing the null that \( H_0 : \hat{\alpha} = \alpha \)
Unconditional Coverage (UC) test

- The violation r.v. $V_{t+1}^{1-\alpha}$ (for $t = 0, \cdots, T - 1$) is a binomial random variable with probability $\alpha$, that is,

$$P(V_{t+1}^{1-\alpha} = 1) = \alpha$$

and

$$P(V_{t+1}^{1-\alpha} = 0) = 1 - \alpha$$

- Example: let’s say that we have only three days and in the second day there was a violation, but not in the other days

- $V_1 = 0$, $V_2 = 1$, $V_3 = 0$ and the probability of this event is

$$P(V_1 = 0, V_2 = 1, V_3 = 0) = (1 - \alpha)^1 \times \alpha^0 \times (1 - \alpha)^0 \times \alpha^1 \times (1 - \alpha)^1 \times \alpha^0$$

so that

$$P(V_1 = 0, V_2 = 1, V_3 = 0) = (1 - \alpha)^2 \alpha^1$$

- More generally, if we have a sample of $T$ days with $T_1$ days with a violation ($V_{t+1}^{1-\alpha} = 1$) and $T_0$ without a violation ($V_{t+1}^{1-\alpha} = 0$), the joint probability of the sequence of violation is given by

$$\mathcal{L}(\alpha, T_1, T) = \alpha^{T_1} (1 - \alpha)^{T_0}$$
The UC test is performed by comparing the likelihood above calculated at the theoretical probability 0.01 with the likelihood calculated at the estimate of $\alpha$, $\hat{\alpha} = T_1 / T$

The statistic and distribution of the test for Unconditional Coverage (UC) are

$$UC = -2 \ln \left( \frac{L(\alpha, T_1, T)}{L(\hat{\alpha}, T_1, T)} \right) \sim \chi^2_1$$

where $\alpha$ is the level at which VaR was calculated, $\hat{\alpha} = T_1 / T$ and $\chi^2_1$ denotes the chi-square distribution with 1 degree-of-freedom.

The critical values at 1, 5, and 10% are 6.63, 3.84, and 2.71, respectively

The null hypothesis $H_0 : \hat{\alpha} = \alpha$ is rejected if $UC$ is larger than the critical value
In practice, the test statistic can be calculated as follows:

\[-2 \left[ T_1 \ln \left( \frac{\alpha}{\hat{\alpha}} \right) + T_0 \ln \left( \frac{1 - \alpha}{1 - \hat{\alpha}} \right) \right]\]

In the example discussed above we have \( \hat{\alpha} = 0.01 \), \( T_1 = 74 \), and \( T \) is 6860. The test statistic is thus 0.42

Since 0.42 is smaller than 3.84 we do not reject at 5% significance level the null hypothesis \( H_0 : \hat{\alpha} = \alpha \) and conclude that the risk model provides appropriate unconditional coverage.
Another property that we would like the risk model to satisfy is that violations happen at random times.

This property does not hold when violations happen in consecutive days.

To test this hypothesis we calculate two quantities:

1. $\alpha_{1,1}$: the probability of a violation in day $t$ given that a violation occurred in day $t-1$ (and $\alpha_{1,0} = 1 - \alpha_{1,1}$ probability of no violation in $t$ given the violation the previous day)

2. $\alpha_{0,1}$: the probability of having a violation in day $t$ given that no violation occurred the previous day (and $\alpha_{0,0} = 1 - \alpha_{0,1}$ )

They can be estimated from the data by calculating $T_{1,1}$ and $T_{0,1}$ that represent the number of days in which a violation was preceded by a violation and a no violation, respectively.
In R we can determine these quantities as follows:

\[
\begin{align*}
T_{11} & = \text{sum}((\text{lag}(V)==1) \& (V==1)) \\
T_{01} & = \text{sum}((\text{lag}(V)==0) \& (V==1))
\end{align*}
\]

The results are:

- \( T_{0,1} = 74 \) and \( T_{0,0} = 6711 \)
- \( T_{1,1} = 0 \) and \( T_{1,0} = 74 \)

The conditional probabilities are then given by:

- \( \hat{\alpha}_{0,1} = T_{0,1}/(T_{0,1} + T_{0,0}) = 0.011 \) and \( \alpha_{0,0} = 1 - \alpha_{0,1} \)
- \( \hat{\alpha}_{1,1} = T_{1,1}/(T_{1,0} + T_{1,1}) = 0 \) and \( \alpha_{1,0} = 1 - \alpha_{1,1} \)
The likelihood of observing $T_{0,1}$ and $T_{1,1}$ sequences out of $T$ is given by

$$L(\hat{\alpha}_{0,1}, \hat{\alpha}_{1,1}, T_{0,1}, T_{1,1}, T) = \hat{\alpha}_{1,0}^{T_{1,0}} (1 - \hat{\alpha}_{1,0})^{T_{1,1}} \hat{\alpha}_{0,1}^{T_{0,1}} (1 - \hat{\alpha}_{0,1})^{T_{0,0}}$$

The **IND** test statistic is calculated as

$$IND = -2 \ln \left( \frac{L(\hat{\alpha}, T_{0}, T)}{L(\hat{\alpha}_{0,1}, \hat{\alpha}_{1,1}, T_{0,1}, T_{1,1}, T)} \right) \sim \chi^2_1$$

where:

- the numerator is the same as the denominator in the UC test and represents the likelihood under independence.
- The logarithm of the denominator is calculated as:

$$T_{1,0} \ln(\alpha_{1,0}) + T_{1,1} \ln(1 - \alpha_{1,0}) + T_{0,1} \ln(\alpha_{0,1}) + T_{0,0} \ln(1 - \alpha_{0,1})$$

The test statistic and distribution for the hypothesis of **Independence** (IND) in this case is 1.61 which is smaller relative to the critical value at 5% and we conclude that we do not reject the null hypothesis $H_0 : \alpha_{0,1} = \alpha_{1,1} = \hat{\alpha}$. The violations are independent.
Conditional coverage

In this case the hypothesis that we are interested in testing is

\[ H_0 : \alpha_{0,1} = \alpha_{1,1} = \alpha \]

**CC** tests jointly independence of the violations and that coverage is always equal to 1%

The test statistic is calculated as

\[ CC = LR^{IND} + LR^{UC} \sim \chi^2_2 \]

The critical value at 5% is 5.99 and the test statistic is 2.03 so that we do not reject the null hypothesis of correct conditional coverage.
Summary on testing risk models

1. Unconditional coverage:
   - $H_0 : \alpha = 0.01$
   - $UC = -2 \ln \left( \frac{\mathcal{L}(0.01, T, T')}{\mathcal{L}(\hat{\alpha}, T, T)} \right)$
   - $\chi^2_1$

2. Independence:
   - $H_0 : \alpha_{0,1} = \alpha_{1,1} = \hat{\alpha}$
   - $IND = -2 \ln \left( \frac{\mathcal{L}(\hat{\alpha}, T_0, T)}{\mathcal{L}(\hat{\alpha}_{0,1}, \hat{\alpha}_{1,1}, T_0, T_1, T)} \right)$
   - $\chi^2_1$

3. Conditional coverage:
   - $H_0 : \alpha_{0,1} = \alpha_{1,1} = 0.01$
   - $CC = UC + IND$
   - $\chi^2_2$

▶ The End!