Rational Shapes of the Volatility Surface

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Merrill Lynch
References


• J. Gatheral, Courant Institute of Mathematical Sciences Lecture Notes, http://www.math.nyu.edu/fellows_fin_math/gatheral/.


Goals

• Derive arbitrage bounds on the slope and curvature of volatility skews.
• Investigate the strike and time behavior of these bounds.
• Specialize to stochastic volatility and jumps.
• Draw implications for parameterization of the volatility surface.
Slope Constraints

• No arbitrage implies that call spreads and put spreads must be non-negative. i.e.
\[
\frac{\partial C}{\partial K} \leq 0 \quad \text{and} \quad \frac{\partial P}{\partial K} \geq 0
\]

• In fact, we can tighten this to
\[
\frac{\partial C}{\partial K} \leq 0 \quad \text{and} \quad \frac{\partial}{\partial K} \left( \frac{P}{K} \right) \geq 0
\]
• Translate these equations into conditions on the implied total volatility $\sigma[y]$ as a function of $y = \ln\left(\frac{K}{F}\right)$.

• In conventional notation, we get

$$
\sigma'[y] \leq \sqrt{2\pi} \exp\left\{d_2^2/2\right\} N(d_2)
$$

$$
\sigma'[y] \geq -\sqrt{2\pi} \exp\left\{d_1^2/2\right\} N(-d_1)
$$
• Assuming $\sigma[y] = 0.25 - 0.3y$ we can plot these bounds on the slope as functions of $y$. 
• Note that we have plotted bounds on the slope of total implied volatility as a function of $y$. This means that the bounds on the slope of BS implied volatility get tighter as time to expiration increases by $1/\sqrt{T}$. 
Convexity Constraints

• No arbitrage implies that call and puts must have positive convexity. *i.e.*

\[ \frac{\partial^2 C}{\partial K^2} \geq 0 \text{ and } \frac{\partial^2 P}{\partial K^2} \geq 0 \]

• Translating these into our variables gives

\[ \frac{\partial^2 C}{\partial y^2} \geq \frac{\partial C}{\partial y} \]
• We get a complicated expression which is nevertheless easy to evaluate for any particular function $\sigma[y]$.

$$\sigma''[y] \geq \frac{1}{4 \sigma[y]^3}$$

$$(4\sigma[y]^2 + 8y\sigma[y] \sigma'[y] - 4y^2 \sigma'[y]^2 + \sigma[y]^4 \sigma'[y]^2)$$

• This expression is equivalent to demanding that butterflies have non-negative value.
• Again, assuming $\sigma[y] = 0.25$ and $\sigma'[y] = -0.3$
  we can plot this lower bound on the convexity as a function of $\sigma$.
Implication for Variance Skew

• Putting together the vertical spread and convexity conditions, it may be shown that implied variance may not grow faster than linearly with the log-strike.
• Formally,

\[
\frac{v[y]}{y} \equiv \frac{\sigma_{BS}^2[y]}{y} \rightarrow \text{some constant } A \text{ as } |y| \rightarrow \infty
\]
Local Volatility

- Local volatility $\sigma(K,T)$ is given by

$$\sigma^2(K,T) = \frac{\partial C}{\partial T} \left( \frac{1}{2} \frac{\partial^2 C}{\partial K^2} \right)$$

- Local variances are non-negative iff arbitrage constraints are satisfied.
Time Behavior of the Skew

• Since in practice, we are interested in the lower bound on the slope for most stocks, let’s investigate the time behavior of this lower bound.

• Recall that the lower bound on the slope can be expressed as

\[-\sqrt{2\pi} \exp\left\{-d_1^2/2\right\} N(-d_1)\]
• For small times, \( d_1 \approx 0 \) and \( N(-d_1) \approx \frac{1}{2} \)

so

\[
\sigma'[0] \geq -\sqrt{\frac{\pi}{2}}
\]

Reinstating explicit dependence on T, we get

\[
\sigma_{BS}'[0] \geq -\sqrt{\frac{\pi}{2T}}
\]

That is, \( \sqrt{T} \) for small \( T \).
• Also,

\[ d_1 = \frac{\sigma[0]}{2} \rightarrow \infty \text{ as } t \rightarrow \infty \]

• Then, the lower bound on the slope

\[ \sigma'[0] \geq -\sqrt{2\pi} \exp\left\{d_1^2/2\right\} N(-d_1) \]

\[ \approx -\frac{1}{d_1} = -\frac{2}{\sigma[0]} \]

• Making the time-dependence of \( \sigma[0] \) explicit,

\[ \sigma_{BS}'[0] \geq -\frac{1}{T} \frac{2}{\sigma_{BS}[0]} \text{ as } T \rightarrow \infty \]
• In particular, the time dependence of the at-the-money skew cannot be

\[ \sigma_{BS}'[0] \approx -\frac{1}{\sqrt{T}} \]

because for any choice of positive constants \( a, b \)

\[ \exists T \text{ large enough s.t. } -\frac{a}{\sqrt{T}} < -\frac{b}{T} \]
• Assuming $\sigma_{BS}[0] = 0.25$, we can plot the variance slope lower bound as a function of time.
A Practical Example of Arbitrage

• We suppose that the ATMF 1 year volatility and skew are 25% and 11% per 10% respectively. Suppose that we extrapolate the vol skew using a $1/\sqrt{T}$ rule.

• Now, buy 99 puts struck at 101 and sell 101 puts struck at 99. What is the value of this portfolio as a function of time to expiration?
| Current Market | 100.00 | 100.00 | 100.00 | 100.00 |
| Dividends (cts. yield or schedule) | 0.00% | 0.00% | 0.00% | 0.00% |
| Strike | 101.00 | 99.00 | 101.00 | 99.00 |
| Start Date (date on which strike is set) | 03-Apr-98 | 03-Apr-98 | 03-Apr-98 | 03-Apr-98 |
| Shares = s, Notional = n | s | s | s | s |
| Expiration Date | 03-Apr-99 | 03-Apr-99 | 03-Apr-02 | 03-Apr-02 |
| Stock Rate (sa/365 rate or curve) | 0.000% | 0.000% | 0.000% | 0.000% |
| Pay Rate (sa/365 rate or curve) | 0.000% | 0.000% | 0.000% | 0.000% |
| Volatility (number or curve) | 23.90% | 26.10% | 24.45% | 25.55% |
| Call =c, Put= p | p | p | p | p |
| Option Price | 10.07 | 9.84 | 19.92 | 19.58 |
| Delta | -0.4690 | -0.4329 | -0.4113 | -0.3916 |
| Gamma (per 1%) | 0.0166 | 0.0151 | 0.0080 | 0.0075 |
| Vega per 1% vol | 0.3976 | 0.3932 | 0.7774 | 0.7675 |
| Theta per day | -0.0130 | -0.0141 | -0.0065 | -0.0067 |
| Position | 99 | -101 | 99 | -101 |
| Value | 996.72 | (993.70) | 1,972.34 | (1,977.18) |
| Portfolio Value | 3.02 | | | (4.83) |

Arbitrage!
With more reasonable parameters, it takes a long time to generate an arbitrage though….

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50 Years!  
No arbitrage!
So Far…. 

- We have derived arbitrage constraints on the slope and convexity of the volatility skew.
- We have demonstrated that the $1/\sqrt{T}$ rule for extrapolating the skew is inconsistent with no arbitrage. Time dependence must be at most $1/T$ for large $T$. 
Stochastic Volatility

• Consider the following special case of the Heston model:

\[ dx = \mu \, dt + \sqrt{v} \, dZ \]
\[ dv = -\lambda (v - \bar{v}) \, dt - \eta \sqrt{v} \, dZ \]

• In this model, it can be shown that

\[ \frac{\partial v_{BS}}{\partial y} \bigg|_{y=0} \approx -\eta \frac{1}{\lambda T} \left\{ 1 - \frac{1 - e^{-\lambda T}}{\lambda T} \right\} \]
• For a general stochastic volatility theory of the form:

\[ dx = \mu \, dt + \sqrt{v} \, dZ_1 \]

\[ dv = -\lambda (v - \bar{v}) \, dt - \eta \beta(v) \sqrt{v} \, dZ_2 \]

with

\[ \langle dZ_1, dZ_2 \rangle = \rho \, dt \]

we claim that (very roughly)

\[
\left. \frac{\partial v_{BS}}{\partial y} \right|_{y=0} \approx \rho \eta \beta(v) \frac{1}{\lambda T} \left\{ 1 - \frac{1 - e^{-\lambda T}}{\lambda T} \right\}
\]
• Then, for very short expirations, we get
  \[ \frac{\partial v_{BS}}{\partial y} \bigg|_{y=0} \approx \frac{\rho \eta \beta(v)}{2} \]
  - a result originally derived by Roger Lee and for very long expirations, we get
  \[ \frac{\partial v_{BS}}{\partial y} \bigg|_{y=0} \approx \frac{\rho \eta \beta(v)}{\lambda T} \]
  • Both of these results are consistent with the arbitrage bounds.
Doesn’t This Contradict $\sqrt{T}$?

- Market practitioners’ rule of thumb is that the skew decays as $1/\sqrt{T}$.
- Using $\lambda = 1.15$ (from Bakshi, Cao and Chen), we get the following graph for the relative size of the at-the-money variance skew:
ATM Skew as a Function of $T$

Relative Skew

-0.8
-1
-1.2
-1.4
-1.6
-1.8
0.5 1 1.5 2 $T$

Stochastic Vol. ($\lambda = 1.15$)

$1/\sqrt{T}$

Actual SPX skew (5/31/00)
Heston Implied Variance

Parameters: $v = 0.04, \bar{v} = 0.04, \lambda = 1.15, \rho = -0.39, \eta = 0.64$

from Bakshi, Cao and Chen.
A Simple Regime Switching Model

• To get intuition for the impact of volatility convexity, we suppose that realised volatility over the life of a one year option can take one of two values each with probability 1/2. The average of these volatilities is 20%.

• The price of an option is just the average option price over the two scenarios.

• We graph the implied volatilities of the resulting option prices.
High Vol: 21%; Low Vol: 19%
High Vol: 39%; Low Vol: 1%
Intuition

- As $|y| \to \infty$, implied volatility tends to the highest volatility.
- If volatility is unbounded, implied volatility must also be unbounded.
- From a trader’s perspective, the more out-of-the-money (OTM) an option is, the more vol convexity it has. Provided volatility is unbounded, more OTM options must command higher implied volatility.
Asymmetric Variance Gamma
Implied Variance

Parameters: \( \bar{w} = 0.04, \nu = 0.1, \theta = -1.5, \rho = -0.4 \)
Jump Diffusion

• Consider the simplest form of Merton’s jump-diffusion model with a constant probability $\lambda$ of a jump to ruin.
• Call options are valued in this model using the Black-Scholes formula with a shifted forward price.
• We graph 1 year implied variance as a function of log-strike with $\nu = 0.04, \lambda = 0.05$:
Jump-to-Ruin Model

Parameters: $\bar{v} = 0.04, \lambda = 0.05$

$y = \ln(K/F)$
• So, even in jump-diffusion, \( v \) is linear in \( y \) as \( |y| \to \infty \).

• In fact, we can show that for many economically reasonable stochastic-volatility-plus-jump models, implied BS variance must be asymptotically linear in the log-strike \( y \) as \( |y| \to \infty \).

• This means that it does not make sense to plot implied BS variance against delta. As an example, consider the following graph of \( v \) vs. \( \delta \) in the Heston model:
Variance vs $\delta$ in the Heston Model

\[ \text{Variance} \]

$\delta$
Implications for Parameterization of the Volatility Surface

• Implied BS variance $\nu$ must be parameterized in terms of the log-strike $y$ ($\nu$ vs delta doesn’t work).

• $\nu$ is asymptotically linear in $y$ as $|y| \to \infty$

\[
\frac{\partial \nu}{\partial y} \bigg|_{y=0} \quad \text{decays as} \quad \frac{1}{T} \quad \text{as} \quad T \to \infty
\]

• $\frac{\partial \nu}{\partial y} \bigg|_{y=0}$ tends to a constant as $T \to 0$