Time Value of Money Summary Notation and Formulae

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1 Commonly used notations

- Present value, $PV$

- Future value, $FV_n$, where the subscript $n$ is used as an indicator for the time of the future, for example, $n$ periods later.

- $m$—some times, we use $n$ to denote number of years and $m$ to denote the number of compounding per year so that $n \times m$ is the number of compounding periods. For example, for monthly compounding for ten years, $m = 12, n = 10$ and the number of compounding periods is $n \times m = 120$. $m$ is 1 for annual compounding, 2 for semi-annual compounding, 4 for quarterly compounding, 12 for monthly compounding, 52 for weekly compounding, and 365 for daily compounding.

- Rate: it links $PV$ and $FV$, and is often denoted as either $r$ or $i$, or $R$. Note that all rates are quoted in annualized (per year) terms. Their difference is compounding frequency.

- In this write-up, I will use $N = n \times m$ to denote number of periods, $r$ as the annualized rate.

- APR: Annual percentage rate. It is an annualized rate quoted by the market. You not only need to know this rate, but also its compounding frequency.

- EAR: Effective annual compounding rate. For any quoted rate at any compounding frequency, you can convert it into effective an annual compounding rate to facilitate comparison.
2 Formulae

- The most basic time value of money formula that links $PV$ with $FV$ is
  \[ FV_N = PV \left( 1 + \frac{r}{m} \right)^{n \cdot m} \]

- One can solve for $PV$ from $FV$:
  \[ PV = \frac{FV_N}{(1 + \frac{r}{m})^{n \cdot m}} \]

- Solving for compounding rate $r$,
  \[ r = \left( \frac{FV_N}{PV} \right)^{\frac{1}{n \cdot m}} - 1 \]

- Solving for number of years $n$,
  \[ n = \frac{1}{m} \ln \left( \frac{FV_N}{PV} \right) \ln \left( 1 + \frac{r}{m} \right) \]

- Converting APR into EAR
  \[ EAR = \left( 1 + \frac{APR}{m} \right)^m - 1 \]

- Converting from one quote to another. Let $R(m_1)$ and $R(m_2)$ denote the rates compounded at two different frequencies, $m_1$ and $m_2$ times per year. For them to be equivalent, they must generate the same future value if the starting present value is the same. Use $PV = 1$ as a normalization, we ask the future value for one year to be equal
  \[ \left( 1 + \frac{R(m_1)}{m_1} \right)^{m_1} = \left( 1 + \frac{R(m_2)}{m_2} \right)^{m_2} \]

Hence,
\[ R(m_1) = m_1 \left( \left[ \left( 1 + \frac{R(m_2)}{m_2} \right)^{m_2} \right]^{\frac{1}{m_1}} - 1 \right). \]

A logically simple way to think about the problem is: If you are given one rate, say, $R(m_2)$, compute the one-year future value of one dollar first, $FV_1 = \left( 1 + \frac{R(m_2)}{m_2} \right)^{m_2}$. Then, you can compute the rate
from the future value based on whatever compounding,

$$R(m_1) = m_1 \left( \left( \frac{FV_1}{PV} \right)^{\frac{1}{m_1}} - 1 \right) = m_1 \left( \left( FV_1 \right)^{\frac{1}{m_1}} - 1 \right).$$

- Continuous compounding: When $m \to \infty$, you compound “infinite” number of times per year, it is called continuous compounding. In this case, the valuation formula becomes

$$FV = PV e^{rn}$$
$$PV = FV e^{-rn}$$
$$r = \frac{1}{n} \ln \left( \frac{FV}{PV} \right)$$
$$n = \frac{1}{r} \ln \left( \frac{FV}{PV} \right),$$
$$EAR = e^{rn} - 1,$$

where $n$ is number of years, $r$ is the continuous compounding rate, and $e$ is the natural number (approximately equal to 2.718282). In your calculator, it may be written as EXP. Remember that compounding frequency is just a quotation method. The rate together with the compounding frequency determines how much money you make, you can always convert the quote into an annual compounding rate (EAR).

- Present value of an ordinary annuity with $N$ payments $C$, ($m$ payments per year for $n$ years)

$$PV = \frac{C}{(r/m)} \left( 1 - \left(1 + \frac{r}{m}\right)^{-nm} \right)$$

- Future value of the ordinary annuity,

$$FV_N = PV \left(1 + \frac{r}{m}\right)^{nm}$$
$$= \frac{C}{(r/m)} \left( \left(1 + \frac{r}{m}\right)^{nm} - 1 \right)$$
• One can solve for the payment $C$ given the present value (or future value) and the rate/compounding/years,

$$
C = \frac{PV \left( \frac{r}{m} \right)}{\left(1 - (1 + \frac{r}{m})^{-n/m}\right)}
$$

$$
C = \frac{FV \left( \frac{r}{m} \right)}{\left( (1 + \frac{r}{m})^{n/m} - 1 \right)}
$$

• One can also solve for the number of periods, given the payment $C$, the interest rate per period ($r$), and present value,

$$
n = -\frac{\ln \left(1 - \frac{PV \cdot r}{C}\right)}{\ln(1 + r)}
$$

• The present value of a perpetuity (annuity with infinite number of payments $N \to \infty$)

$$
PV = \frac{C}{(r/m)}
$$

3 Examples

1. When you graduate three years from now, you expect to find a job that pays $80,000 per year and you expect your pay to go up 10% per year. According to your expectation, how much money you will be making at your ten-year working anniversary?

• **Answer:** You are essentially asked to compound $80,000 for 10 years at 10% annual compounding rate. It does not matter when you graduate because the question asks about 10 years after your graduation. The answer is $80 \times (1 + 0.1)^{10} = 207.50$ thousands

• In applying the TVM formula, you can regard $80000$ as $PV$ (present value, starting value) and regarding the question as asking for $FV_{10} = PV \left(1 + r\right)^{10}$. Of course, you can also regard $80,000$ as $FV_{3}$ (future value 3 years later) and you are solving for $FV_{13} = FV_{3} \left(1 + r\right)^{13-3}$. What matters is the length of time (number of years in this case) in between starting and ending.

2. Reverse the question: If you expect to make $100,000 at your 10-year working anniversary and you expect your pay to grow at 10% per year, what should be your initial salary?
3. You want to make a saving today and let it to grow to 100,000 in ten years. How much do you need to save if

(a) you can make 10% per year annual compounding? – **Answer:** You are computing the present value of a future value: \( PV = \frac{FV}{(1+r)^t} = \frac{100000}{(1.1)^{10}} = 38,554. \)

(b) a bank is willing to pay you 9.8% monthly compounding? – **Answer:** \( PV = \frac{100000}{(1+\frac{0.098}{12})^{12*10}} = 37681. \)

(c) Assuming the two investment opportunities have the same risk (or no risk at all), which one should you take? 10% annual compounding or 9.8% monthly compounding? – **Answer:** There are several ways to look at this problem: (1) Since you want to reach the same target, whichever asks you to pay less now is the better choice. Hence, it is the bank’s 9.8% monthly compounding offer that is better because you just need to put in 37681 today instead of 38554 to get 100,000 ten years later. This means that although 9.8% is a lower return number, but the faster compounding frequency more than makes up for it. (2) Hence, in comparing returns, you must convert them into the same compounding frequency before you can compare them. The tradition/convention is to convert everything into annual compounding rates: 10% annual compounding is still 10% annual compounding. To convert the 9.8% monthly compounding rate into an annual compounding rate, we have

\[
\left( \frac{1}{\frac{12}{1}} \right) - 1 = 10.252%,
\]

which is higher (and hence better) than 10%. The textbook call this annual compounding rate as **Effective Annual Rate (EAR)** and the quoted rate as **Quoted Annual Rate (APR)**. In this example, the quoted rate (APR) is 9.8% monthly compounding, and we can convert it to an EAR as 10.252%. The conversion is

\[
EAR = \left( 1 + \frac{APR}{m} \right)^m - 1, \tag{1}
\]
where \( m \) is number of compounding per year for the APR.

(d) It is important to understand the origin of this conversion: If you want to convert everything to annual compounding return (EAR), all you need to make sure is that they make the same amount of money over the same time interval, hence the equality:

\[
P\left(1+\frac{APR}{m}\right)^{mn} = PV \left(1+EAR\right)^n,
\]

where \( n \) denotes \( n \) years into the future. The equality says that starting from the same PV, whether you use APR rate at your \( m \)-times per year compounding or use EAR at annual compounding, you should be making the same amount of future value for any \( n \) years down the road. Then, you can see \( PV \) and \( n \) cancels out, and you can get to the formula: \( EAR = \left(1 + \frac{APR}{m}\right)^m \).

(e) Note that the conversion works even if the quoted rate compounds slower than once per year. For example, if the quoted rate compounds once every two years, \( m = 1/2 \) (once every two years).

4. A newly elected President claims that he will strive to double the GDP (gross domestic output) during his 8-year terms (yes, he wants to be re-elected). What annual compounding GDP growth rate he needs in order to reach his target?

- **Answer:** We don’t know exactly what the current or future GDP is, but we know \( FV_8/PV = 2 \) (FV is twice as much as PV). Use the TVM formula, \( FV_8 = PV \left(1+r\right)^8 \), we have

\[
r = \left(\frac{FV_8}{PV}\right)^{1/8} - 1 = 2^{1/8} - 1 = 9.05%.
\]

- Similar questions: A fund/bank/somebody offers to pay you back twice as much in 10 years, what kind of return you are getting if the return is compounded annually, quarterly, or weekly?
  - **Answer:** The general equation is

\[
r = \left[\left(\frac{FV}{PV}\right)^{\frac{1}{n}} - 1\right]^m,
\]

with \( n \) number of years and \( m \) number of compounding per year. For the rate per period, all
you need is number of periods $n \ast m$, but then you need to convert it into an annual rate. The solutions are

$$\text{Annual: } 2^{1/10} - 1 = 7.18\%$$

$$\text{Quarterly: } \left(2^{1/40} - 1\right) \ast 4 = 6.99\%$$

$$\text{Weekly: } \left(2^{1/520} - 1\right) \ast 52 = 6.94\%.$$  

You need a lower return if the compounding frequency is higher.

5. If your investment makes 10% per year annual compounding, how many years does it take to double?

- **Answer**: $PV=1, FV=2, r = 10, m = 1$,

$$n = \frac{\ln (FV/PV)}{\ln (1+r)} = \frac{\ln 2}{\ln (1+0.1)} = 7.2725\text{ years.}$$

- How many years does it take if the 10% rate is compounded weekly? –**Answer**: $m = 52$ and

$$n = \frac{1}{m} \frac{\ln (FV/PV)}{\ln (1+r/m)} = \frac{1}{52} \frac{\ln 2}{\ln (1+0.1/52)} = 6.938\text{ years.}$$

At the same quoted rate, compounding faster makes more money and reaches your goal earlier.

6. If you make a monthly saving of $1000 and earn a 10% monthly compounding rate of return, how much money will you have in your account after 10 years of consecutive savings?

- **Answer**: 10% is per year (annualized rate) even though it is monthly compounding. The question asks for the FV of an annuity,

$$FV = \frac{1000}{0.1/12} \left(\left(1 + \frac{0.1}{12}\right)^{120} - 1\right) = 2.0484 \times 10^5.$$  

- If you start savings 10 years from now, this is how much you get 20 years from now (that is, after you save for 10 years or 120 monthly savings). As long as the question is asking for the value
at your last saving, it does not matter when it starts, it just matters on how many consecutive savings you have made by then.

7. You need to take a $500,000 mortgage to buy a house. The bank charges you a 6% monthly compounding rate for a 30-year loan of $500,000 that pays every month for 360 months. What will be your monthly payment?

- **Answer:** You know $PV$ of the annuity and are asked to solve for the payment,

$$C = \frac{PV \left( \frac{L}{m} \right)}{\left( 1 - \left( 1 + \frac{r}{m} \right)^{-n \times m} \right)} = \frac{500000 \times \frac{0.06}{12}}{\left( 1 - \left( 1 + \frac{0.06}{12} \right)^{-360} \right)} = 2997.8$$

The monthly payment is about $3000.

- The $3000 monthly payment covers both the amount you borrow ($500,000) and the 6% interest the bank charges you. In the U.S., interest payment is tax deductible but the principal payment (payment for the $500,000) is not. So you want to figure out: Out of the $3000 monthly payment, which part is considered as interest payment, and which part covers your loan. This separation is also important in case you want to payback everything in the middle–Then, you want to figure out how much you still owe.

- The calculation starts as follows:

  (a) Figure out how much you still owe the bank (your remaining debt). Initially, you owe them $500,000, but as you pay them back, your debt will reduce.

  (b) Your remaining debt times (6%/12 = 0.005) (monthly rate) is how much you owe them in interest–This is your interest payment.

  (c) Whatever left in your $3000 payment reduces your debt.

Let me use a table to show the separation:
<table>
<thead>
<tr>
<th>Month</th>
<th>Debt Before Payment</th>
<th>Interest Expense</th>
<th>Debt Reduction</th>
<th>Debt After Payment</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>500000</td>
<td>500000 × 0.005 = 2500.0</td>
<td>3000 - 2500 = 500</td>
<td>500000 - 500 = 499500</td>
</tr>
<tr>
<td>2</td>
<td>499500</td>
<td>499500 × 0.005 = 2497.5</td>
<td>3000 - 2497.5 = 502.5</td>
<td>499500 - 502.5 = 498,997.50</td>
</tr>
<tr>
<td>3</td>
<td>498,997.50</td>
<td>2492.99</td>
<td>505.01</td>
<td>498,492.49</td>
</tr>
<tr>
<td>4</td>
<td>498,492.49</td>
<td>2492.46</td>
<td>507.54</td>
<td>497,984.87</td>
</tr>
<tr>
<td>5</td>
<td>...</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

- Suppose the bank gives you the loan today, but “kindly” allows you start your first payment five years later instead of right now, how will your monthly payment change? — **Answer:** The bank is still charging you 6% rate, so there is nothing “kind” about it. If you start late, you need to pay extra interest for the five years you delay back. Hence, a $500,000 loan today becomes a $500000 \times (1 + \frac{0.06}{12})^{5 \times 12} = 6.7443 \times 10^5$ debt 5 years down the road. Then, you are making monthly payment on an annuity that has a present value (at the start of the annuity) of $674,430. The monthly payment increases accordingly to,

\[
C = \frac{500000 \times (1 + \frac{0.06}{12})^{5 \times 12} \times \frac{0.06}{12}}{\left(1 - (1 + \frac{0.06}{12})^{-360}\right)} = 4043.5,
\]

which can also be computed directly from the original monthly payment: \(2997.8 \left(1 + \frac{0.06}{12}\right)^{12 \times 5} = 4043.5\). The idea is that either you can start to make the monthl payment now or you can delay the payment for five years, but you need to pay interest for your delay on each payment, which is LaterPayment=CurrentPayment\left(1 + \frac{0.06}{12}\right)^{12 \times 5}.

8. An apartment’s rental is $3000 per month. Suppose the rent is fixed forever. At a 6% monthly compounding rate, what’s the fair value for this apartment?

- **Answer:** People disagree on what constitutes “fair.” It could be based on cost or demand. In our case, let’s define fair relative to the rent. That is, we want to know how much all these rent payments are worth in today’s dollar terms. We are then essentially calculating the present value of a perpetuity,

\[
PV = \frac{3000}{(0.06/12)} = 600,000.
\]
This is a quick way to get a rough idea on how much an apartment is worth: 200 times monthly rent.

9. You are evaluating an investment project that is projected to generate $100 million in one year, $300mm in year 2, $700mm in year 3, and $−200 in year 4 (The negative cash outflow is because you need to pay to clean up the site, for example). At a 10% annual compounding return, please compute how much this project is worth today.

- The question asks you to compute the PV of multiple, irregular cashflows. You just need to compute the PV of each cash flow and sum them together:

\[
PV = \frac{100}{(1 + 0.1)^1} + \frac{300}{(1 + 0.1)^2} + \frac{700}{(1 + 0.1)^3} + \frac{-200}{(1 + 0.1)^4} = 728.16\text{mm}
\]

The project is worth $728.16 million in today’s dollar term.

- How much is this project worth in two years later’s dollar? —Answer: Just compound the PV for two years,

\[
FV_2 = PV (1 + 0.1)^2 = 728.16 \times (1 + .1)^2 = 881.07.
\]

You can also directly convert each cash flow into year-2 dollar:

\[
FV_2 = 100(1 + 0.1)^1 + 300 + \frac{700}{(1 + 0.1)^1} + \frac{-200}{(1 + 0.1)^2} = 881.07.
\]

To convert year-1 dollar (100) into year-2 dollar, you need to compound one year. To convert year-3 dollar (700) into year-2 dollar, you need discount one year. To convert year-4 dollar (-200), you need to discount two years.

10. Let me use a series of questions to highlight the idea that with a fixed rate, time value comes only in relative terms (the number of years between the conversion), not in absolute terms (e.g., 2014 or 2054).

- Suppose you plan to invest $100 in a fund 10 years later and expect to make 10% a year annual
compounding, how much will your investment grow into one year after your investment? How will your answer change if you decide to make the investment 5 years later, or 50 years later?

**Answer:** You invest $100 for one year at 10% rate and you will end up with $100(1+0.1)=$110. Since the question is asking you the value ”one year after your investment,” it does not matter when (which year) you make the investment. What matters is that your investment will make one year of return.

- Suppose you plan to make an annual saving of $500 for 10 consecutive years for an annual compounding return of 6%. You want to know how much money you will have at the time of the last saving. Say you plan to make your first saving 15 years from now. How much will you have at the end of the last saving?

**Answer:** Again, the answer does not depend on whether you start your saving today or 5, 10, or 15 years later because the question asks you about the value of your savings at the end of your saving. It is the same as the previous question except here we have multiple cash flows/savings. This is a 10-year annuity and you want to know its future value:

\[
FV = \frac{C}{r} \left( (1 + r)^{10} - 1 \right) = \frac{500}{0.06} \left( 1.06^{10} - 1 \right) = 6590.4
\]

- Your friend wants to borrow $2000 from you now and promises to pay you back in 10 consecutive annual payments of $500 each, with the first payment made 15 years from today. If you want a 6% annual compounding return, are you willing to lend him $2000 today?

**Answer:** At 6% discount rate, you want to figure out whether the present value of the 10 payments of $500 each is worth $2000. So you are computing the present value of an annuity. The annuity is the same as in the previous question. This time, when the first payment is made matters because you are computing its today’s value and you have to measure the time distance between today and the payment time. There are a few ways of doing this:

(a) From the answer to the previous question, you know its future value \( FV = 6590.4 \). This is the value at the end of the last payment, which is 15+9=24 years from now — Count carefully! I always make mistake here. The first payment is in year 15. The 10th payment
is in year 24. So you need to discount it back for 24 years

\[ PV = \frac{6590.4}{(1 + 0.06)^{24}} = 1627.7 \]

The present value is only 1627.7, not worth $2000. So you should not lend him $2000 if you want 6% return.

(b) Use the present value of annuity formula. Since the first payment is in year 15. If you look at the problem in year 14, it is an ordinary annuity. You can use the present value of annuity formula to solve the value in year 14 and then discount it back to today.

\[
\text{Value in year 14} = PV \text{ of annuity} = \frac{C}{r} \left(1 - \left(1 + r\right)^{-n}\right) = \frac{500}{0.06} \left(1 - 1.06^{-10}\right) = 3680.0
\]

\[
PV = \frac{\text{Value in year 14}}{(1 + 0.06)^{14}} = \frac{3680}{1.06^{14}} = 1627.7.
\]