The Black-Scholes Model

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(Continuous time finance primer)
Outline

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The Black-Scholes-Merton (BSM) model

- Black and Scholes (1973) and Merton (1973) derive option prices under the following assumption on the stock price dynamics,

\[ dS_t = \mu S_t dt + \sigma S_t dW_t \]

- The binomial model: Discrete states and discrete time (The number of possible stock prices and time steps are both finite).

- The BSM model: Continuous states (stock price can be anything between 0 and \( \infty \)) and continuous time (time goes continuously).

- Scholes and Merton won Nobel price. Black passed away.

- BSM proposed the model for stock option pricing. Later, the model has been extended/twisted to price currency options (Garman&Kohlhagen) and options on futures (Black).

- I treat all these variations as the same concept and call them indiscriminately the BSM model.
Primer on continuous time process

\[ dS_t = \mu S_t dt + \sigma S_t dW_t \]

- The driver of the process is \( W_t \), a Brownian motion, or a Wiener process.
- \( W_t \) generates random “continuous movements:” Movements arrive continuously, & the sizes of the movements are random and center around zero.
- Contrast: Merton (1976) proposes a stock price process with both continuous movements (\( W_t \)) and compound Poisson jumps: The jump arrives randomly over time (say on average once every two years). Given one jump occurring, the jump size is randomly distributed.
- It is appropriate to use Brownian motion to capture daily random price movements, and use the compound Poisson jumps to rare but extreme events (such as market crashes, credit events, etc).
Properties of a Brownian motion

\[ dS_t = \mu S_t \, dt + \sigma S_t \, dW_t \]

- The process \( W_t \) generates a random variable that is normally distributed with mean 0 and variance \( t \), \( \phi(0, t) \).

- The process is composed of independent normal increments \( dW_t \sim \phi(0, dt) \).

- “\( d \)” is the continuous time-time limit of the discrete time difference (\( \Delta \)).
- \( \Delta t \) denotes a finite time step (say, 3 months), \( dt \) denotes an extremely thin slice of time (smaller than 1 milisecond).
- It is so thin that it is often referred to as *instantaneous*.
- Similarly, \( dW_t = W_{t+dt} - W_t \) denotes the instantaneous increment (change) of a Brownian motion.

- By extension, increments over non-overlapping time periods are independent: For \( t_1 > t_2 > t_3 \), \( (W_{t_3} - W_{t_2}) \sim \phi(0, t_3 - t_2) \) is independent of \( (W_{t_2} - W_{t_1}) \sim \phi(0, t_2 - t_1) \).
Properties of a normally distributed random variable

\[ dS_t = \mu S_t dt + \sigma S_t dW_t \]

- If \( X \sim \phi(0, 1) \), then \( a + bX \sim \phi(a, b^2) \).
- If \( y \sim \phi(m, V) \), then \( a + by \sim \phi(a + bm, b^2 V) \).
- Since \( dW_t \sim \phi(0, dt) \), the instantaneous price change 
  \[ dS_t = \mu S_t dt + \sigma S_t dW_t \sim \phi(\mu S_t dt, \sigma^2 S_t^2 dt). \]
- The instantaneous return 
  \[ \frac{dS}{S} = \mu dt + \sigma dW_t \sim \phi(\mu dt, \sigma^2 dt). \]
- The annualized instantaneous return 
  \[ \frac{1}{dt} \left[ \frac{dS}{S} \right] \sim \phi(\mu, \sigma^2). \]
  - Under the BSM model, \( \mu \) is the annualized mean of the instantaneous return — instantaneous mean return.
  - \( \sigma^2 \) is the annualized variance of the instantaneous return — instantaneous return variance.
  - \( \sigma \) is the annualized standard deviation of the instantaneous return — instantaneous return volatility.
Geometric Brownian motion

\[ \frac{dS_t}{S_t} = \mu dt + \sigma dW_t \]

- The stock price is said to follow a \textit{geometric} Brownian motion.
- \( \mu \) is often referred to as the \textit{drift}, and \( \sigma \) the \textit{diffusion} of the process.
- Instantaneously, the stock price change is normally distributed, \( \phi(\mu S_t dt, \sigma^2 S_t^2 dt) \).
- Over longer horizons, the price change is \textit{lognormally} distributed.
- The log return (continuous compounded return) is normally distributed over all horizons:
\[
d\ln S_t = (\mu - \frac{1}{2} \sigma^2) dt + \sigma dW_t. \text{ (By Ito’s lemma, chap 11).}
\]
  - \( d\ln S_t \sim \phi(\mu dt - \frac{1}{2} \sigma^2 dt, \sigma^2 dt) \).
  - \( \ln S_t \sim \phi(\ln S_0 + \mu t - \frac{1}{2} \sigma^2 t, \sigma^2 t) \).
  - \( \ln S_T / S_t \sim \phi(\left( \mu - \frac{1}{2} \sigma^2 \right) (T - t), \sigma^2 (T - t)) \).
- Integral form: \( S_t = S_0 e^{\mu t - \frac{1}{2} \sigma^2 t + \sigma W_t} \), \( \ln S_t = \ln S_0 + \mu t - \frac{1}{2} \sigma^2 t + \sigma W_t \).
Aggregating normally distributed random variables

- If \( y_i \) is independent of one another and \( y_i \sim \phi(m, V) \) for all \( i \), then
  \[ \sum_{i=1}^{N} y_i \sim \phi(Nm, NV). \]

- Mean and variance increase with time periods \( N \), volatility increases with square root of time periods, \( \sqrt{N} \).

- Application: Instantaneous return \( d \ln S_t \sim \phi(\mu dt - \frac{1}{2} \sigma^2 dt, \sigma^2 dt) \) has identical and independent normal distribution (iid normal), the aggregated return over finite time period (say, \( \ln S_T / S_t \)) remains normal with mean and variance proportional to the time period \( (T - t) \).

- If \( y_i \) is independent but not identical: \( y_i \sim \phi(m_i, V_i) \) with \( (m_i, V_i) \), the distribution of \( \sum_{i=1}^{N} y_i \) is unknown (no longer normal).
  
  - Example: The instantaneous price change \( dS_t \sim \phi(\mu S_t dt, \sigma^2 S_t^2 dt) \), independent but not identical (it varies with \( S_t \) level). The aggregated price change is no longer normally distributed.
Examples on time aggregation

- Suppose daily returns are iid normal with mean 0.05% and standard deviation is 1.5%. What is the distribution of the aggregated return over 1 year (252 business days)?
  - The aggregated 1-year return is also normally distributed, and has a mean of $0.05\% \times 252 = 12.6\%$.
  - The annual return has a variance of $(1.5\%)^2 \times 252 = 0.0567$.
  - *The annual return has a volatility of* $1.5\% \times \sqrt{252} = 23.81\%$.

- Pay attention to the aggregation on volatility.
  - Another example: Suppose the annual return volatility for a stock is 20%, under the iid normal assumption, what’s the monthly return volatility?

- Remember: $W_t$ has a volatility of $\sqrt{t}$.
  - Which of the following pairs are identical in distribution?
    (A) $(2W_t, W_{2t})$, (B) $(4W_t, W_{2t})$, (C) $(4 + 4W_t, 2 + W_{2t})$, (D) $(4 + 9W_t, 4 + W_{3t})$. 

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Normal versus lognormal distribution

\[ dS_t = \mu S_t dt + \sigma S_t dW_t, \quad \mu = 10\%, \sigma = 20\%, S_0 = 100, t = 1. \]

The earliest application of Brownian motion to finance is Louis Bachelier in his dissertation (1900) “Theory of Speculation.” He specified the stock price as following a Brownian motion with drift:

\[ dS_t = \mu dt + \sigma dW_t \]
Simulate 100 stock price sample paths

\[ dS_t = \mu S_t dt + \sigma S_t dW_t, \quad \mu = 10\%, \sigma = 20\%, S_0 = 100, t = 1. \]

- Stock with the return process: \( d \ln S_t = (\mu - \frac{1}{2} \sigma^2) dt + \sigma dW_t. \)
- Discretize to daily intervals \( dt \approx \Delta t = 1/252. \)
- Draw standard normal random variables \( \varepsilon(100 \times 252) \sim \phi(0, 1). \)
- Convert them into daily log returns: \( R_d = (\mu - \frac{1}{2} \sigma^2) \Delta t + \sigma \sqrt{\Delta t} \varepsilon. \)
- Convert returns into stock price sample paths: \( S_t = S_0 e^{\sum_{d=1}^{252} R_d}. \)
\[ \mu \text{ versus } \mu - \frac{1}{2} \sigma^2 \]

Suppose we have daily data for a period of several months

- **Arithmetic mean**: If you compute daily percentage return \( \Delta S / S \), and take the simple sample average on the return. After annualization, it should be close to \( \mu \).

  \[ \hat{\mu} = \frac{252}{N} \left[ \frac{1}{N} \sum_{d=1}^{N} \frac{\Delta S_d}{S_d} \right] \]

- **Geometric mean**: If you start at day 1 and compound continuously (or daily, it should be close), the return per year is close to \( \mu - \frac{1}{2} \sigma \).

  - **Daily**: \( \frac{252}{N} \left\{ \left[ \left( 1 + \frac{\Delta S_1}{S_1} \right) \left( 1 + \frac{\Delta S_2}{S_2} \right) \cdots \left( 1 + \frac{\Delta S_N}{S_N} \right) \right]^{1/N} - 1 \right\} \)
  
  - **Continuous**: \( \frac{252}{N} \left\{ \frac{1}{N} \ln \left[ \frac{S_1}{S_0} \frac{S_2}{S_1} \cdots \frac{S_N}{S_{N-1}} \right] \right\} = \frac{252}{N} \left\{ \ln \left[ \frac{S_N}{S_0} \right] \right\} \)
Example

Suppose that returns in successive years are: 15%, 20%, 30%, -20% and 25%.

- **Arithmetic mean**: \[
  \frac{0.15 + 0.20 + 0.30 - 0.20 + 0.25}{5} = 0.14.
\]

- **Geometric mean**: \[
  \left[(1 + 0.15)(1 + 0.20)(1 + 0.30)(1 - 0.20)(1 + 0.25)\right]^{1/5} - 1 = 0.124.
\]

**Comments:**

- Geometric mean is normally lower than arithmetic mean due to something called “concavity (convexity) adjustment.”
- If \( R_t \sim \phi(m, V) \), then \( E[R_t] = m \), but

  \[
  E[e^{R_t}] = e^{m + \frac{1}{2}V}, \quad \ln E[e^{R_t}] = m + \frac{1}{2}V
  \]
Ito’s lemma on continuous processes

For a generic continuous process $x_t$, 

$$dx_t = \mu_x dt + \sigma_x dW_t,$$

the transformed variable $y = f(x, t)$ follows,

$$dy_t = \left(f_t + f_x \mu_x + \frac{1}{2} f_{xx} \sigma_x^2\right) dt + f_x \sigma_x dW_t$$

- **Example 1:** $dS_t = \mu S_t dt + \sigma S_t dW_t$ and $y = \ln S_t$. We have $d \ln S_t = \left(\mu - \frac{1}{2} \sigma^2\right) dt + \sigma dW_t$

- **Example 2:** $dv_t = \kappa (\theta - v_t) dt + \omega \sqrt{v_t} dW_t$, and $\sigma_t = \sqrt{v_t}$.

  $$d\sigma_t = \left(\frac{1}{2} \sigma_t^{-1} \kappa \theta - \frac{1}{2} \kappa \sigma_t - \frac{1}{8} \omega^2 \sigma_t^{-1}\right) dt + \frac{1}{2} \omega dW_t$$

- **Example 3:** $dx_t = -\kappa x_t dt + \sigma dW_t$ and $v_t = a + bx_t^2$. $\Rightarrow dv_t = \text{?}$
Ito’s lemma on jumps

For a generic process $x_t$ with jumps,

$$dx_t = \mu_x dt + \sigma_x dW_t + \int g(z) (\mu(dz, dt) - \nu(z, t) dz dt),$$

$$= (\mu_x - \mathbb{E}_t[g]) dt + \sigma_x dW_t + \int g(z) \mu(dz, dt),$$

where the random counting measure $\mu(dz, dt)$ realizes to a nonzero value for a given $z$ if and only if $x$ jumps from $x_{t-}$ to $x_t = x_{t-} + g(z)$ at time $t$. The process $\nu(z, t)dzdt$ compensates the jump process so that the last term is the increment of a pure jump martingale. The transformed variable $y = f(x, t)$ follows,

$$dy_t = \left( f_t + f_x \mu_x + \frac{1}{2} f_{xx} \sigma_x^2 \right) dt + f_x \sigma_x dW_t$$
$$+ \int (f(x_{t-} + g(z), t) - f(x_{t-}, t)) \mu(dz, dt) - \int f_x g(z) \nu(z, t) dz dt,$$

$$= \left( f_t + f_x (\mu_x - \mathbb{E}_t[g]) + \frac{1}{2} f_{xx} \sigma_x^2 \right) dt + f_x \sigma_x dW_t$$
$$+ \int (f(x_{t-} + g(z), t) - f(x_{t-}, t)) \mu(dz, dt).$$
Ito’s lemma on jumps: Merton (1976) example

\[
dy_t = (f_t + f_x \mu_x + \frac{1}{2} f_{xx} \sigma_x^2) \, dt + f_x \sigma_x \, dW_t \\
+ \int (f(x_{t-} + g(z), t) - f(x_{t-}, t)) \mu(dz, dt) - \int f_x g(z) \nu(z, t) \, dz \, dt,
\]

\[
= (f_t + f_x (\mu_x - \mathbb{E}_t[g(z)]) + \frac{1}{2} f_{xx} \sigma_x^2) \, dt + f_x \sigma_x \, dW_t \\
+ \int (f(x_{t-} + g(z), t) - f(x_{t-}, t)) \mu(dz, dt)
\]

Merton (1976)’s jump-diffusion process:

- \[dS_t = \mu S_t \, dt + \sigma S_t \, dW_t + \int S_t (e^z - 1) (\mu(z, t) - \nu(z, t) \, dz \, dt), \text{ and} \]
- \[y = \ln S_t. \text{ We have} \]

\[
d \ln S_t = \left( \mu - \frac{1}{2} \sigma^2 \right) \, dt + \sigma \, dW_t + \int z \mu(dz, dt) - \int (e^z - 1) \nu(z, t) \, dz \, dt
\]

- \[f(x_{t-} + g(z), t) - f(x_{t-}, t) = \ln S_t - \ln S_{t-} = \ln(S_{t-} e^z) - \ln S_{t-} = z. \]

- The specification \((e^z - 1)\) on price guarantees that the maximum downside jump is no larger than the pre-jump price level \(S_{t-}\).

- In Merton, \[\nu(z, t) = \lambda \frac{1}{\sigma_j \sqrt{2\pi}} e^{-\frac{(z-\mu_j)^2}{2\sigma_j^2}}. \]
The key idea behind BSM

- The option price and the stock price depend on the same underlying source of uncertainty.
- The Brownian motion dynamics implies that if we slice the time thin enough (\(dt\)), it behaves like a binomial tree.
- Reversely, if we cut \(\Delta t\) small enough and add enough nodes, the binomial tree converges to the distribution behavior of the geometric Brownian motion.

  - Under this thin slice of time interval, we can combine the option with the stock to form a riskfree portfolio.
  - Recall our hedging argument: Choose \(\Delta\) such that \(f - \Delta S\) is riskfree.
  - The portfolio is riskless (under this thin slice of time interval) and must earn the riskfree rate.
  - Magic: \(\mu\) does not matter for this portfolio and hence does not matter for the option valuation. Only \(\sigma\) matters.
The hedging proof

- The stock price dynamics: $dS = \mu Sdt + \sigma SdW_t$.

- Let $f(S, t)$ denote the value of a derivative on the stock, by Ito’s lemma: $df_t = \left( f_t + f_S \mu S + \frac{1}{2} f_{SS} \sigma^2 S^2 \right) dt + f_S \sigma SdW_t$.

- At time $t$, form a portfolio that contains 1 derivative contract and $-\Delta = f_S$ of the stock. The value of the portfolio is $P = f - f_S S$.

- The instantaneous uncertainty of the portfolio is $f_S \sigma SdW_t - f_S \sigma SdW_t = 0$. Hence, instantaneously the delta-hedged portfolio is riskfree.

- Then, the portfolio must earn the riskfree rate: $dP = rPdt$.

- $dP = df - f_S dS = \left( f_t + f_S \mu S + \frac{1}{2} f_{SS} \sigma^2 S^2 - f_S \mu S \right) dt = r(f - f_S S)dt$

- Hence, the fundamental partial differential equation: $f_t + rSf_S + \frac{1}{2} f_{SS} \sigma^2 S^2 = rf$

- No where do we see the drift of the price dynamics ($\mu$).
The hedging argument leads to the following partial differential equation:

\[
\frac{\partial f}{\partial t} + (r - q)S \frac{\partial f}{\partial S} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 f}{\partial S^2} = rf
\]

- The only free parameter is \( \sigma \) (as in the binominal model).

Solving this PDE, subject to the terminal payoff condition of the derivative (e.g., \( f_T = (S_T - K)^+ \) for a European call option), BSM derive analytical formulas for call and put option value.

- Similar formula had been derived before based on distributional (normal return) argument, but \( \mu \) was still in.
- The realization that option valuation does not depend on \( \mu \) is big. Plus, it provides a way to hedge the option position.

The PDE is generic for any derivative securities, as long as \( S \) follows geometric Brownian motion.

- Given boundary conditions, derivative values can be solved numerically from the PDE.
Explicit finite difference method

- One way to solve the PDE numerically is to discretize across time using $N$ time steps $(0, \Delta t, 2\Delta t, \cdots, T)$ and discretize across states using $M$ grids $(0, \Delta S, \cdots, S_{\text{max}})$.

- We can approximate the partial derivatives at time $i$ and state $j$, $(i, j)$, by the differences:
  
  \[
  f_S \approx \frac{f_{i,j+1} - f_{i,j-1}}{2\Delta S}, \quad f_{SS} \approx \frac{f_{i,j+1} + f_{i,j-1} - 2f_{i,j}}{\Delta S^2}, \quad \text{and} \quad f_t = \frac{f_{i,j} - f_{i-1,j}}{\Delta t}
  \]

- The PDE becomes:
  
  \[
  \frac{f_{i,j} - f_{i-1,j}}{\Delta t} + (r - q)j\Delta S \frac{f_{i,j+1} - f_{i,j-1}}{2\Delta S} + \frac{1}{2} \sigma^2 j^2 (\Delta S)^2 = rf_{i,j}
  \]

- Collecting terms:
  \[
  f_{i-1,j} = a_j f_{i,j-1} + b_j f_{i,j} + c_j f_{i,j+1}
  \]

  - Essentially a trinominal tree: The time-$i$ value at $j$ state of $f_{i,j}$ is a weighted average of the time-$i+1$ values at the three adjacent states $(j-1, j, j+1)$. 

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The Black-Scholes Model

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Finite difference methods: variations

- At \((i,j)\), we can define the difference \((f_t, f_S)\) in three different ways:
  
  ▶ Backward:  \( f_t = (f_{i,j} - f_{i-1,j})/\Delta t \).
  
  ▶ Forward:  \( f_t = (f_{i+1,j} - f_{i,j})/\Delta t \).
  
  ▶ Centered:  \( f_t = (f_{i+1,j} - f_{i-1,j})/(2\Delta t) \).

- Different definitions results in different numerical schemes:
  
  ▶ Explicit:  \( f_{i-1,j} = a_j f_{i,j-1} + b_j f_{i,j} + c_j f_{i,j+1} \).
  
  ▶ Implicit:  \( a_j f_{i,j-1} + b_j f_{i,j} + c_j f_{i,j+1} = f_{i+1,j} \).
  
  ▶ Crank-Nicolson:  
    \[ -\alpha_j f_{i-1,j-1} + (1 - \beta_j) f_{i-1,j} - \gamma_j f_{i-1,j+1} = \alpha_j f_{i,j-1} + (1 + \beta_j) f_{i,j} + \gamma_j f_{i,j+1}. \]
The BSM formula for European options

\[
\begin{align*}
    c_t &= S_t e^{-q(T-t)} N(d_1) - Ke^{-r(T-t)} N(d_2), \\
p_t &= -S_t e^{-q(T-t)} N(-d_1) + Ke^{-r(T-t)} N(-d_2),
\end{align*}
\]

where

\[
\begin{align*}
    d_1 &= \frac{\ln(S_t/K) + (r-q)(T-t) + \frac{1}{2} \sigma^2(T-t)}{\sigma \sqrt{T-t}}, \\
    d_2 &= \frac{\ln(S_t/K) + (r-q)(T-t) - \frac{1}{2} \sigma^2(T-t)}{\sigma \sqrt{T-t}} = d_1 - \sigma \sqrt{T-t}.
\end{align*}
\]

Black derived a variant of the formula for futures (which I like better):

\[
c_t = e^{-r(T-t)} [F_t N(d_1) - KN(d_2)],
\]

with \(d_{1,2} = \frac{\ln(F_t/K) \pm \frac{1}{2} \sigma^2(T-t)}{\sigma \sqrt{T-t}}\).

- Recall: \(F_t = S_t e^{(r-q)(T-t)}\).
- Once I know call value, I can obtain put value via put-call parity:
  \(c_t - p_t = e^{-r(T-t)} [F_t - K_t]\).
Cumulative normal distribution

\[ c_t = e^{-r(T-t)} \left[ F_t N(d_1) - KN(d_2) \right], \quad d_{1,2} = \frac{\ln(F_t/K) \pm \frac{1}{2}\sigma^2(T-t)}{\sigma\sqrt{T-t}} \]

- \(N(x)\) denotes the cumulative normal distribution, which measures the probability that a normally distributed variable with a mean of zero and a standard deviation of 1 (\(\phi(0,1)\)) is less than \(x\).

- Most software packages (including excel) has efficient ways to computing this function.

Properties of the BSM formula:

- As \(S_t\) becomes very large or \(K\) becomes very small, \(\ln(F_t/K) \uparrow \infty, N(d_1) = N(d_2) = 1\). \(c_t = e^{-r(T-t)} \left[ F_t - K \right]\).

- Similarly, as \(S_t\) becomes very small or \(K\) becomes very large, \(\ln(F_t/K) \uparrow -\infty, N(-d_1) = N(-d_2) = 1\). \(p_t = e^{-r(T-t)} \left[ -F_t + K \right]\).
Implied volatility

\[ c_t = e^{-r(T-t)} [F_t N(d_1) - KN(d_2)], \quad d_{1,2} = \frac{\ln(F_t/K) \pm \frac{1}{2}\sigma^2(T-t)}{\sigma \sqrt{T-t}} \]

- Since \( F_t \) (or \( S_t \)) is observable from the underlying stock or futures market, \((K, t, T)\) are specified in the contract. The only unknown (and hence free) parameter is \( \sigma \).

- We can estimate \( \sigma \) from time series return. (standard deviation calculation).

- Alternatively, we can choose \( \sigma \) to match the observed option price — implied volatility (IV).

- There is a one-to-one correspondence between prices and implied volatilities.

- Traders and brokers often quote implied volatilities rather than dollar prices.

- The BSM model says that \( IV = \sigma \). In reality, the implied volatility calculated from different options (across strikes, maturities, dates) are usually different.
Why does it matter?

- As long as we assume that the underlying security price follows a geometric Brownian motion, we can use (some versions) of the BSM formula to price European options.

- Dividends, foreign interest rates, and other types of carrying costs may complicate the pricing formula a little bit.

- A simpler approach: Assume that the underlying futures/forwards price (of the same maturity of course) process follows a geometric Brownian motion.

- Then, as long as we observe the forward price (or we can derive the forward price), we do not need to worry about dividends or foreign interest rates — They are all accounted for in the forward pricing.

- Know how to price a forward, and use the Black formula.
Risk-neutral valuation

- Recall: Under the binomial model, we derive a set of risk-neutral probabilities such that we can calculate the expected payoff from the option and discount them using the risk-free rate.
  - Risk premiums are hidden in the risk-neutral probabilities.
  - If in the real world, people are indeed risk-neutral, the risk-neutral probabilities are the same as the real-world probabilities. Otherwise, they are different.

- Under the BSM model, we can also assume that there exists such an artificial risk-neutral world, in which the expected returns on all assets earn risk-free rate.

- The stock price dynamics under the risk-neutral world becomes,
  \[ \frac{dS_t}{S_t} = (r - q)dt + \sigma dW_t. \]

- Simply replace the actual expected return (\(\mu\)) with the return from a risk-neutral world (\(r - q\)) \([\text{ex-dividend return}]\).

- We label the true probability measure by \(\mathbb{P}\) and the risk-neutral measure by \(\mathbb{Q}\).
Readings behind the technical jargons: P v. Q

- **P**: Actual probabilities that earnings will be high or low, estimated based on historical data and other insights about the company.
  - Valuation is all about getting the forecasts right and assigning the appropriate price for the forecasted risk — *fair wrt future cashflows (and your risk preference)*.

- **Q**: “Risk-neutral” probabilities that we can use to aggregate expected future payoffs and discount them back with riskfree rate, regardless of how risky the cash flow is.
  - It is related to real-time scenarios, but it has nothing to do with real-time probability.
  - Since the intention is to hedge away risk under all scenarios and discount back with riskfree rate, we do not really care about the actual probability of each scenario happening. We just care about what all the possible scenarios are and whether our hedging works under all scenarios.
  - **Q** is not about getting close to the actual probability, but about being *fair wrt the prices of securities that you use for hedging*.
The risk-neutral return on spots

\[
\frac{dS_t}{S_t} = (r - q)dt + \sigma dW_t, \text{ under risk-neutral probabilities.}
\]

- In the risk-neutral world, investing in all securities make the riskfree rate as the total return.
- If a stock pays a dividend yield of \( q \), then the risk-neutral expected return from stock price appreciation is \( r - q \), such as the total expected return is: dividend yield + price appreciation = \( r \).
- Investing in a currency earns the foreign interest rate \( r_f \) similar to dividend yield. Hence, the risk-neutral expected currency appreciation is \( r - r_f \) so that the total expected return is still \( r \).
- Regard \( q \) as \( r_f \) and value options as if they are the same.
If we sign a forward contract, we do not pay anything upfront and we do not receive anything in the middle (no dividends or foreign interest rates). Any P&L at expiry is excess return.

Under the risk-neutral world, we do not make any excess return. Hence, the forward price dynamics has zero mean (driftless) under the risk-neutral probabilities: $\frac{dF_t}{F_t} = \sigma dW_t$.

The carrying costs are all hidden under the forward price, making the pricing equations simpler.
Return predictability and option pricing

Consider the following stock price dynamics,

\[
\begin{align*}
  \frac{dS_t}{S_t} &= (a + bX_t) \, dt + \sigma dW_t, \\
  dX_t &= \kappa (\theta - X_t) + \sigma_x dW_{2t}, \quad \rho dt = \mathbb{E}[dW_t dW_{2t}].
\end{align*}
\]

Stock returns are predictable by a vector of mean-reverting predictors \( X_t \). *How should we price options on this predictable stock?*

- Option pricing is based on replication/hedging — a cross-sectional focus, not based on prediction — a time series behavior.
- Whether we can predict the stock price is irrelevant. The key is whether we can replicate or hedge the option risk using the underlying stock.
- Consider a derivative \( f(S, t) \), whose terminal payoff depends on \( S \) but not on \( X \), then it remains true that \( df_t = \left( f_t + f_S \mu S + \frac{1}{2} f_{SS} \sigma^2 S^2 \right) \, dt + f_S \sigma S dW_t \).
- The delta-hedged option portfolio \(( f - f_S S)\) remains riskfree.
- The same PDE holds for \( f \):
  \[
  f_t + r S f_S + \frac{1}{2} f_{SS} \sigma^2 S^2 = rf
  \]
Return predictability and risk-neutral valuation

Consider the following stock price dynamics (under $\mathbb{P}$),

\[
\begin{align*}
dS_t/S_t &= (a + bX_t) \, dt + \sigma \, dW_t, \\
dX_t &= \kappa (\theta - X_t) + \sigma_x \, dW_{2t}, \quad \rho dt = \mathbb{E}[dW_t \, dW_{2t}].
\end{align*}
\]

- Under the risk-neutral measure ($\mathbb{Q}$), stocks earn the riskfree rate, regardless of the predictability: $dS_t/S_t = (r - q) \, dt + \sigma \, dW_t$

- Analogously, the futures price is a martingale under $\mathbb{Q}$, regardless of whether you can predict the futures movements or not: $dF_t/F_t = \sigma \, dW_t$

- Measure change from $\mathbb{P}$ to $\mathbb{Q}$ does not change the diffusion component $\sigma \, dW$.

- The BSM formula still applies.
Stochastic volatility and option pricing

Consider the following stock price dynamics,

\[
\frac{dS_t}{S_t} = \mu dt + \sqrt{\nu_t} dW_t, \\
\frac{dv_t}{\nu_t} = \kappa (\theta - v_t) + \omega \sqrt{\nu_t} dW_{2t}, \\
\rho dt = \mathbb{E}[dW_t dW_{2t}].
\]

**How should we price options on this stock with stochastic volatility?**

- Consider a call option on \( S \). Even though the terminal value does not depend on \( \nu_t \), the value of the option at other periods have to depend on \( \nu_t \).
  - Assume \( f(S, t) \) does not depend on \( \nu_t \), we have
    \[
    df_t = \left( f_t + f_S \mu S + \frac{1}{2} f_{SS} \nu_t S^2 \right) dt + f_S \sigma S \sqrt{\nu_t} dW_t,
    \]
    which obviously depend on \( \nu_t \).
  - For \( f(S, v, t) \), we have (bivariate version of Ito):
    \[
    df_t = \left( f_t + f_S \mu S + \frac{1}{2} f_{SS} \nu_t S^2 + f_v \kappa (\theta - \nu_t) + \frac{1}{2} f_{vv} \omega^2 \nu_t + f_{sv} \omega \nu_t \rho \right) dt + f_S \sigma S \sqrt{\nu_t} dW_t + f_v \omega \sqrt{\nu_t} dW_{2t}.
    \]
- Since there are two sources of risk \( (W_t, W_{2t}) \), delta-hedging is not going to lead to a riskfree portfolio.
Pricing kernel

- In the absence of arbitrage, in an economy, there exists at least one strictly positive process, $M_t$, the state-price deflator, such that the deflated gains process associated with any admissible strategy is a martingale. The ratio of $M_t$ at two horizons, $M_{t,T}$, is called a stochastic discount factor, or more informally, a pricing kernel.

  ▶ For an asset that has a terminal payoff $\Pi_T$, its time-$t$ value is $p_t = \mathbb{E}_t^P [M_{t,T}\Pi_T]$.
  ▶ The time-$t$ price of a $1$ par riskfree zero-coupon bond that expires at $T$ is $p_t = \mathbb{E}_t^P [M_{t,T}]$.
  ▶ In a discrete-time representative agent economy with an additive CRRA utility, the pricing kernel is equal to the ratio of the marginal utilities of consumption,

$$M_{t,t+1} = \beta \frac{u'(c_{t+1})}{u'(c_t)} = \beta \left( \frac{c_{t+1}}{c_t} \right)^{-\gamma}$$
From pricing kernel to exchange rates

- Let $M_{t,T}^h$ denote the pricing kernel in economy $h$ that prices all securities in that economy with its currency denomination.

- The $h$-currency price of currency-$f$ ($h$ is home currency) is linked to the pricing kernels of the two economies by,

$$
\frac{S_{t}^{fh}}{S_{T}^{fh}} = \frac{M_{t}^f}{M_{t}^h}
$$

- Hence, the log currency return over period $[t, T]$, $\ln S_{T}^{fh}/S_{t}^{fh}$ is linked to the difference between the log pricing kernel of the two economies.

- Let $S$ denote the dollar price of pound (e.g. $S_t = $2.06), then

$$
\ln S_T/S_t = \ln M_t^{\text{pound}} - \ln M_t^{\text{dollar}}.
$$

- If markets are completed by primary securities (e.g., bonds and stocks), there is one unique pricing kernel per economy. The exchange rate movement is uniquely determined by the ratio of the pricing kernels.

- If the markets are not completed by primary securities, exchange rates (and currency options) help complete the markets by requiring that the ratio of any two viable pricing kernels must go through the exchange rate.
From pricing kernel to measure change

- In continuous time, it is convenient to define the pricing kernel via the following multiplicative decomposition:

\[ M_{t,T} = \exp \left( - \int_t^T r_s ds \right) \mathcal{E} \left( - \int_t^T \gamma_s^T dX_s \right) \]

- \( r \) is the instantaneous riskfree rate, \( \mathcal{E} \) is the stochastic exponential martingale operator, \( X \) denotes the risk sources in the economy, and \( \gamma \) is the market price of the risk \( X \).

- If \( X_t = W_t \), \( \mathcal{E} \left( - \int_t^T \gamma_s dW_s \right) = \exp \left( - \int_t^T \gamma_s dW_s - \frac{1}{2} \int_t^T \gamma_s^2 ds \right) \).

- In a continuous time version of the representative agent example, \( dX_s = d\ln c_t \) and \( \gamma \) is relative risk aversion.

- \( r \) is usually a function of \( X \).

- The exponential martingale \( \mathcal{E}(\cdot) \) defines the measure change from \( \mathbb{P} \) to \( \mathbb{Q} \).
Measure change defined by exponential martingales

Consider the following exponential martingale that defines the measure change from $\mathbb{P}$ to $\mathbb{Q}$:

$$
\left. \frac{d\mathbb{Q}}{d\mathbb{P}} \right|_t = \mathcal{E} \left( - \int_0^t \gamma_s dW_s \right),
$$

- If the $\mathbb{P}$ dynamics is: $dS_t^i = \mu_t^i S_t^i dt + \sigma_t^i S_t^i dW_t^i$, with $\rho_t^i dt = \mathbb{E}[dW_t dW_t^i]$, then the dynamics of $S_t^i$ under $\mathbb{Q}$ is: $dS_t^i = \mu_t^i S_t^i dt + \sigma_t^i S_t^i dW_t^i + \mathbb{E}[\gamma_t dW_t, \sigma_t^i S_t^i dW_t^i] = (\mu_t^i - \gamma_t \sigma_t^i \rho_t^i) S_t^i dt + \sigma_t^i S_t^i dW_t^i$
- If $S_t^i$ is the price of a traded security, we need $r = \mu_t^i - \gamma_t \sigma_t^i \rho_t^i$. The risk premium on the security is $\mu_t^i - r = \gamma_t \sigma_t^i \rho_t^i$.

How do things change if the pricing kernel is given by:

$$
M_{t,T} = \exp \left( - \int_t^T r_s ds \right) \mathcal{E} \left( - \int_t^T \gamma_s^\top dW_s \right) \mathcal{E} \left( - \int_t^T \gamma_s^\top dZ_s \right),
$$

where $Z_t$ is another Brownian motion independent of $W_t$ or $W_t^i$. 
Bond pricing under $\mathbb{P}$ and $\mathbb{Q}$

- The time-$t$ price of a $1$ par riskfree zero-coupon bond that expires at $T$ is:

$$p_t = \mathbb{E}^\mathbb{P}_t[M_{t,T}]$$
$$= \mathbb{E}^\mathbb{P}_t\left[\exp\left(-\int_t^T r_s ds\right) \mathcal{E} \left(-\int_t^T \gamma_s^\top dX_s\right)\right]$$
$$= \mathbb{E}^\mathbb{Q}_t\left[\exp\left(-\int_t^T r_s ds\right)\right].$$

- We need to know the short rate dynamics under $\mathbb{Q}$ for bond pricing.

- Suppose the pricing kernel is:

$$M_{t,T} = \exp\left(-\int_t^T r_s ds\right) \mathcal{E} \left(-\int_t^T \gamma_s^\top dX_s\right) \mathcal{E} \left(-\int_t^T \eta_s^\top dZ_s\right)$$

with $X, Z$ orthogonal, and $r = r(X)$. Bond price will only depend on $X$, not $Z$. 
Let $X$ denote a pure jump process with its compensator being $\nu(x, t)$ under $\mathbb{P}$.

Consider a measure change defined by the exponential martingale:
\[
\frac{d\mathbb{Q}}{d\mathbb{P}}|_t = \mathcal{E}(\gamma X_t),
\]

The compensator of the jump process under $\mathbb{Q}$ becomes:
\[
\nu(x, t)^Q = e^{-\gamma x} \nu(x, t).
\]

Example: Merton (176)'s compound Poisson jump process,
\[
\nu(x, t) = \lambda \frac{1}{\sigma_j \sqrt{2\pi}} e^{-\frac{(x-\mu_J)^2}{2\sigma_j^2}}. \quad \text{Under } \mathbb{Q}, \text{ it becomes}
\]
\[
\nu(x, t)^Q = \lambda \frac{1}{\sigma_j \sqrt{2\pi}} e^{-\gamma x - \frac{(x-\mu_J)^2}{2\sigma_j^2}} = \lambda^Q \frac{1}{\sigma_j \sqrt{2\pi}} e^{-\frac{(x-\mu_J^Q)^2}{2\sigma_j^2}} \quad \text{with } \mu^Q = \mu_J - \gamma \sigma_j^2
\]
and $\lambda^Q = \lambda e^{\frac{1}{2} \gamma (\gamma \sigma_j^2 - 2\mu_J)}$.

The BSM Delta

The BSM delta of European options (*Can you derive them?*):

\[
\Delta_c \equiv \frac{\partial c_t}{\partial S_t} = e^{-qT} N(d_1), \quad \Delta_p \equiv \frac{\partial p_t}{\partial S_t} = -e^{-qT} N(-d_1)
\]

\[(S_t = 100, T - t = 1, \sigma = 20\%)
\]

- Industry quotes the delta in absolute percentage terms (right panel).
- *Which of the following is out-of-the-money?* (i) 25-delta call, (ii) 25-delta put, (iii) 75-delta call, (iv) 75-delta put.
- The strike of a 25-delta call is close to the strike of: (i) 25-delta put, (ii) 50-delta put, (iii) 75-delta put.
Delta as a moneyness measure

Different ways of measuring moneyness:

- $K$ (relative to $S$ or $F$): Raw measure, not comparable across different stocks.
- $K/F$: better scaling than $K - F$.
- $\ln K/F$: more symmetric under BSM.
- $\frac{\ln K/F}{\sigma \sqrt{(T-t)}}$: standardized by volatility and option maturity, comparable across stocks. Need to decide what $\sigma$ to use (ATMV, IV, 1).
- $d_1$: a standardized variable.
- $d_2$: Under BSM, this variable is the truly standardized normal variable with $\phi(0,1)$ under the risk-neutral measure.

**delta**: Used frequently in the industry

- Measures moneyness: Approximately the percentage chance the option will be in the money at expiry.
- Reveals your underlying exposure (how many shares needed to achieve delta-neutral).
Delta hedging

Example: A bank has sold for $300,000 a European call option on 100,000 shares of a nondividend paying stock, with the following information: $S_t = 49, K = 50, r = 5\%, \sigma = 20\%, (T - t) = 20\text{weeks}, \mu = 13\%$.

- What’s the BSM value for the option? $\rightarrow$ $2.4$
- What’s the BSM delta for the option? $\rightarrow$ $0.5216$.

Strategies:

- **Naked position**: Take no position in the underlying.
- **Covered position**: Buy 100,000 shares of the underlying.
- **Stop-loss strategy**: Buy 100,000 shares as soon as price reaches $50$, sell 100,000 shares as soon as price falls below $50$.
- **Delta hedging**: Buy 52,000 share of the underlying stock now. Adjust the shares over time to maintain delta-neutral.
  - Need frequent rebalancing (daily) to maintain delta neutral.
  - Involves a “buy high and sell low” trading rule.
Delta hedging with futures

- The delta of a futures contract is $e^{(r-q)(T-t)}$.
- The delta of the option with respect to (wrt) futures is the delta of the option over the delta of the futures.
- The delta of the option wrt futures (of the same maturity) is
  \[
  \Delta_{c/F} \equiv \frac{\partial c_t}{\partial F_{t,T}} = \frac{\partial c_t/\partial S_t}{\partial F_{t,T}/\partial S_t} = e^{-rT}N(d_1),
  \]
  \[
  \Delta_{p/F} \equiv \frac{\partial p_t}{\partial F_{t,T}} = \frac{\partial p_t/\partial S_t}{\partial F_{t,T}/\partial S_t} = -e^{-rT}N(-d_1).
  \]
- Whenever available (such as on indexes, commodities), using futures to delta hedge can potentially reduce transaction costs.
OTC quoting and trading conventions for currency options

- Options are quoted at fixed time-to-maturity (not fixed expiry date).

- Options at each maturity are not quoted in invoice prices (dollars), but in the following format:
  - Delta-neutral straddle implied volatility (ATMV):
    A straddle is a portfolio of a call & a put at the same strike. The strike here is set to make the portfolio delta-neutral \( d_1 = 0 \).
  - 25-delta risk reversal: \( IV(\Delta_c = 25) - IV(\Delta_p = 25) \).
  - 25-delta butterfly spreads: \( (IV(\Delta_c = 25) + IV(\Delta_p = 25))/2 - ATMV \).
  - Risk reversals and butterfly spreads at other deltas, e.g., 10-delta.

- When trading, invoice prices and strikes are calculated based on the BSM formula.

- The two parties exchange both the option and the underlying delta.
  - The trades are delta-neutral.
The BSM vega

- Vega ($\nu$) is the rate of change of the value of a derivatives portfolio with respect to volatility — it is a measure of the volatility exposure.

- BSM vega: the same for call and put options of the same maturity

$$\nu = \frac{\partial c_t}{\partial \sigma} = \frac{\partial p_t}{\partial \sigma} = S_t e^{-q(T-t)} \sqrt{T-t} \cdot n(d_1)$$

$n(d_1)$ is the standard normal probability density: $n(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}$.

$$(S_t = 100, T - t = 1, \sigma = 20\%)$$

Volatility exposure (vega) is higher for at-the-money options.
Vega hedging

- Delta can be changed by taking a position in the underlying.
- To adjust the volatility exposure (vega), it is necessary to take a position in an option or other derivatives.

Hedging in practice:

- Traders usually ensure that their portfolios are delta-neutral at least once a day.
- Whenever the opportunity arises, they improve/manage their vega exposure — options trading is more expensive.
- As portfolio becomes larger, hedging becomes less expensive.

- Under the assumption of BSM, vega hedging is not necessary: $\sigma$ does not change. But in reality, it does.
  - Vega hedge is outside the BSM model.
Example: Delta and vega hedging

Consider an option portfolio that is delta-neutral but with a vega of \(-8,000\). We plan to make the portfolio both delta and vega neutral using two instruments:

- The underlying stock
- A traded option with delta 0.6 and vega 2.0.

*How many shares of the underlying stock and the traded option contracts do we need?*

- To achieve vega neutral, we need long \(8000/2=4,000\) contracts of the traded option.
- With the traded option added to the portfolio, the delta of the portfolio increases from 0 to \(0.6 \times 4,000 = 2,400\).
- We hence also need to short 2,400 shares of the underlying stock \(\Rightarrow\) each share of the stock has a delta of one.
Another example: Delta and vega hedging

Consider an option portfolio with a delta of 2,000 and vega of 60,000. We plan to make the portfolio both delta and vega neutral using:

- The underlying stock
- A traded option with delta 0.5 and vega 10.

**How many shares of the underlying stock and the traded option contracts do we need?**

- As before, it is easier to take care of the vega first and then worry about the delta using stocks.
- To achieve vega neutral, we need short/write $60000/10 = 6000$ contracts of the traded option.
- With the traded option position added to the portfolio, the delta of the portfolio becomes $2000 - 0.5 \times 6000 = -1000$.
- We hence also need to long 1000 shares of the underlying stock.
A more formal setup

Let \((\Delta_p, \Delta_1, \Delta_2)\) denote the delta of the existing portfolio and the two hedging instruments. Let \((\nu_p, \nu_1, \nu_2)\) denote their vega. Let \((n_1, n_2)\) denote the shares of the two instruments needed to achieve the target delta and vega exposure \((\Delta_T, \nu_T)\). We have

\[
\Delta_T = \Delta_p + n_1 \Delta_1 + n_2 \Delta_2
\]
\[
\nu_T = \nu_p + n_1 \nu_1 + n_2 \nu_2
\]

We can solve the two unknowns \((n_1, n_2)\) from the two equations.

- **Example 1:** The stock has delta of 1 and zero vega.
  
  \[
  0 = 0 + n_1 \cdot 0.6 + n_2
  \]
  \[
  0 = -8000 + n_1 \cdot 2 + 0
  \]
  
  \[
  n_1 = 4000, \quad n_2 = -0.6 \times 4000 = -2400.
  \]

- **Example 2:** The stock has delta of 1 and zero vega.
  
  \[
  0 = 2000 + n_1 \cdot 0.5 + n_2, \quad 0 = 60000 + n_1 \cdot 10 + 0
  \]
  
  \[
  n_1 = -6000, \quad n_2 = 1000.
  \]

- **When do you want to have non-zero target exposures?**
BSM gamma

- Gamma ($\Gamma$) is the rate of change of delta ($\Delta$) with respect to the price of the underlying asset.

- The BSM gamma is the same for calls and puts:

$$\Gamma \equiv \frac{\partial^2 c_t}{\partial S_t^2} = \frac{\partial \Delta_t}{\partial S_t} = \frac{e^{-q(T-t)}n(d_1)}{S_t\sigma\sqrt{T-t}}$$

$$\left( S_t = 100, \ T - t = 1, \ \sigma = 20\% \right)$$

Gamma is high for near-the-money options. High gamma implies high variation in delta, and hence more frequent rebalancing to maintain low delta exposure.
Gamma hedging

- High gamma implies high variation in delta, and hence more frequent rebalancing to maintain low delta exposure.

- Delta hedging is based on small moves during a very short time period.
  - assuming that the relation between option and the stock is linear locally.

- When gamma is high,
  - The relation is more curved (convex) than linear,
  - The P&L (hedging error) is more likely to be large in the presence of large moves.

- The gamma of a stock is zero.

- We can use traded options to adjust the gamma of a portfolio, similar to what we have done to vega.

- But if we are really concerned about large moves, we may want to try something else.
Dynamic hedging with greeks

The idea of delta and vega hedging is based on a locally linear approximation (partial derivative) of the relation between the derivative portfolio value and the underlying stock price and volatility.

Since the relation is not linear, the hedging ratios change as the environment change.

- **Dynamic hedging**, hedging based on partial derivatives, often asks for frequent rebalancing.

- Dynamic hedging works well if

  The overall relation is close to linear. Hence, the hedging ratio is stable over time. The underlying variable (stock price, volatility) varies smoothly and only changes a little within a certain time interval.
Dynamic versus static hedging

- Dynamic hedging can generate large hedging errors when the underlying variable (stock price) can jump randomly.
  - A large move size per se is not an issue, as long as we know how much it moves — a binomial tree can be very large moves, but delta hedge works perfectly.
  - As long as we know the magnitude, hedging is relatively easy.
  - The key problem comes from large moves of random size.

- An alternative is to devise static hedging strategies: The position of the hedging instruments does not vary over time.
  - Conceptually not as easy. Different derivative products ask for different static strategies.
  - It involves more option positions. Cost per transaction is high.
  - Monitoring cost is low. Fewer transactions.
Example: Static hedging long-term option using short-term options


- A European call option maturing at a fixed time $T$ can be replicated by a static position in a continuum of European call options at a shorter maturity $u \leq T$:

$$C(S, t; K, T) = \int_{0}^{\infty} w(K) C(S, t; K, u) dK,$$

with the weight of the portfolio given by $w(K) = \frac{\partial^2}{\partial K^2} C(K, u; K, T)$, the gamma that the target call option will have at time $u$, should the underlying price level be at $K$ then.

- Assumption: The stock price can jump randomly. The only requirement is that the stock price is Markovian in itself: $S_t$ captures all the information about the stock up to time $t$. 
## Practical implementation and performance

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Liuren Wu  
*The Black-Scholes Model*  
*Option Pricing, Fall, 2007*  
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Semi-static hedge of barriers

- Option being hedged: a one-month one-touch barrier option with a lower barrier and payment at expiry
- Hedging instruments: vanilla and binary put options, binary call.
- Procedure:
  - At time $t$, sell the barrier, and put on a hedging position with vanilla options with the terminal payoff
    \[ 1(S_T < L) \left(1 + \frac{S_T}{L}\right) = 2(S < L) - (L - S)^+ / L \]
  - If the barrier $L$ never hits before expiry, both the barrier and the hedging portfolio generate zero payoff.
  - If the barrier is hit before expiry, then:
    - Sell a European option with the terminal payoff
      \[ 1(S_T < L) \left(\frac{S_T}{L}\right) = (S < L) - (L - S)^+ / L. \]
    - Buy a European option that pays $1(S_T > L)$.
    - The operation is self-financing under the BSM when $r = q$. 
Hedging barriers: Dynamic and static hedging performance

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JPYUSD:

Liuren Wu
### GBPUSD:

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