Using Lévy Processes to Model Return Innovations

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Fall, 2007
Outline

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Lévy processes

- A Lévy process is a continuous-time process that generates stationary, independent increments ...

- Think of return innovations \((\varepsilon)\) in discrete time: \(R_{t+1} = \mu_t + \sigma_t \varepsilon_{t+1}\).
  - Normal return innovation — diffusion
  - Non-normal return innovation — jumps

- Classic Lévy specifications in finance:
  - Brownian motion (Black-Scholes, Merton)
  - Compound Poisson process with normal jump size (Merton)

\(\Rightarrow\) The return innovation distribution is either normal or mixture of normals.
Lévy characteristics

- Lévy processes greatly expand our continuous-time choices of iid return innovation distributions via the Lévy triplet \((\mu, \sigma, \pi(x))\). \((\pi(x)\text{–Lévy density})\).

- The Lévy-Khintchine Theorem:

\[
\begin{align*}
\phi_{X_t}(u) & \equiv \mathbb{E}[e^{iuX_t}] = e^{-t\psi(u)}, \\
\psi(u) & = -iu\mu + \frac{1}{2}u^2\sigma^2 + \int_{\mathbb{R}_0} (1 - e^{iu} + iux1_{|x|<1}) \pi(x)dx,
\end{align*}
\]

Innovation distribution

\( \leftrightarrow \) characteristic exponent \(\psi(u)\)

\( \leftrightarrow \) Lévy triplet \((\mu, \sigma, \pi(x))\)

- Constraint: \(\int_0^1 x^2\pi(x)dx < \infty\).

- “Tractable:” if the integral can be carried out explicitly.

- When well-defined, it is convenient to define the cumulant exponent:

\[
\kappa(s) \equiv \frac{1}{t} \ln \mathbb{E}[e^{sX_t}] = s\mu + \frac{1}{2}s^2\sigma^2 + \int_{\mathbb{R}_0} (e^{sx} - 1 - sx1_{|x|<1}) \pi(x)dx.
\]

\[
\psi(u) = -\kappa(iu), \quad \kappa(s) = -\psi(-is).
\]
Model stock returns with Lévy processes

- Let $X_t$ be a Lévy process, $\kappa_X(s)$ its cumulant exponent.
- The log return on a security can be modeled as

$$\ln S_t/S_0 = \mu t + X_t - t\kappa_X(1)$$

where $\mu$ is the instantaneous drift (mean) of the stock such that $E[S_t] = S_0 e^{\mu t}$. The last term $-t\kappa_X(1)$ is a convexity adjustment such that $X_t - t\kappa_X(1)$ forms an exponential martingale:

$$E[e^{X_t - t\kappa_X(1)}] = 1.$$

- Since both $\mu$ and $\kappa_X(1)$ are deterministic components, they can be combined together: $\ln S_t/S_0 = mt + X_t$, but it is more convenient to separate them so that the mean instantaneous return $\mu$ is kept as a separate free parameter.
- Under $Q$, $\mu = r - q$.
- Under this specification, we shall always set the first component of the Lévy triplet to zero $(0, \sigma, \pi(x))$, because it will be canceled out with the convexity adjustment.
Characteristic function of the security return

\[ s_t \equiv \ln S_t / S_0 = \mu t + X_t - t \kappa_X(1) \]

- The characteristic function for the security return is

\[ \phi_{s_t}(u) \equiv \mathbb{E} \left[ e^{iu \ln S_t / S_0} \right] = \exp \left( - \left[ -iu \mu + \psi_X(u) + iu \kappa_X(1) \right] t \right) \]

- The characteristic exponent is

\[ \psi_{s_t}(u) = -iu \mu + \psi_X(u) + iu \kappa_X(1) \]

- Under \( \mathbb{Q} \), \( \mu = r - q \). The focus of the model specification is on \( X_t \sim (0, \sigma, \pi(x)) \), unless \( r \) and/or \( q \) are modeled to be stochastic.
Tractable examples of Lévy processes

1. Brownian motion (BSM) ($\mu t + \sigma W_t$): normal shocks.

2. Compound Poisson jumps (Merton, 76): Large but rare events.

$$\pi(x) = \lambda \frac{1}{\sqrt{2\pi \nu_J}} \exp \left( -\frac{(x - \mu_J)^2}{2\nu_J} \right).$$

3. Dampened power law (DPL):

$$\pi(x) = \begin{cases} 
\lambda \exp (-\beta_+ x) x^{-\alpha-1}, & x > 0, \\
\lambda \exp (-\beta_- |x|) |x|^{-\alpha-1}, & x < 0,
\end{cases}$$

- **Finite activity** when $\alpha < 0$: $\int_{\mathbb{R}^0} \pi(x) dx < \infty$. Compound Poisson. Large and rare events.
- **Infinite activity** when $\alpha \geq 0$: Both small and large jumps. Jump frequency increases with declining jump size, and approaches infinity as $x \to 0$.
- **Infinite variation** when $\alpha \geq 1$: many small jumps.

*Market movements of all magnitudes, from small movements to market crashes.*
Analytical characteristic exponents

- Diffusion: \( \psi(u) = -iu \mu + \frac{1}{2} u^2 \sigma^2. \)

- Merton’s compound Poisson jumps:
  \[
  \psi(u) = \lambda \left( 1 - e^{i u \mu J - \frac{1}{2} u^2 v J} \right).
  \]

- Dampened power law: (for \( \alpha \neq 0, 1 \))
  \[
  \psi(u) = -\lambda \Gamma(-\alpha) \left[ (\beta_+ - iu)^\alpha - \beta_\alpha^\alpha + (\beta_- + iu)^\alpha - \beta_-^\alpha \right] - iuC(h)
  \]
  ▶ When \( \alpha \to 2 \), smooth transition to diffusion (quadratic function of \( u \)).
  ▶ When \( \alpha = 0 \) (Variance-gamma by Madan et al):
    \[
    \psi(u) = \lambda \ln \left( 1 - \frac{i u}{\beta_+} \right) \left( 1 + \frac{i u}{\beta_-} \right) = \lambda \left( \ln(\beta_+ - iu) - \ln \beta + \ln(\beta_- + iu) - \ln \beta_- \right).
    \]
  ▶ When \( \alpha = 1 \) (exponentially dampened Cauchy, Wu 2006):
    \[
    \psi(u) = -\lambda \left( (\beta_+ - iu) \ln(\beta_+ - iu) / \beta_+ + \lambda (\beta_- + iu) \ln(\beta_- + iu) / \beta_- \right) - iuC(h).
    \]
The Black-Scholes model

- The driver is a Brownian motion $X_t = \sigma W_t$.
- We can write the return as
  \[
  \ln \frac{S_t}{S_0} = \mu t + \sigma W_t - \frac{1}{2} \sigma^2 t.
  \]

Note that $\kappa(s) = \frac{1}{2} s^2 \sigma^2$.

- The characteristic function of the return is:
  \[
  \phi(u) = \exp \left( iu \mu t - \frac{1}{2} u^2 \sigma^2 t - iu \frac{1}{2} \sigma^2 \right) = \exp \left( iu \mu t - \frac{1}{2} \sigma^2 (u^2 + iu) t \right).
  \]

- Under $\mathbb{Q}$, $\mu = r - q$.

- The characteristic exponent of the convexity adjusted Lévy process $(X_t - \kappa X(1)t)$ is: $\psi_X(u) + iu\kappa_X(1) = \frac{1}{2} u^2 \sigma^2 + iu \frac{1}{2} \sigma^2 = \frac{1}{2} \sigma^2 (u^2 + iu)$. 
Merton (1976)'s jump-diffusion model

- The driver of this model is a Lévy process that has both a diffusion component and a jump component.

- The Lévy triplet is $(0, \sigma, \pi(x))$, with $\pi(x) = \lambda \frac{1}{\sqrt{2\pi v_J}} \exp\left(-\frac{(x-\mu_J)^2}{2v_J}\right)$.
  - The first component of the triplet (the drift) is always normalized to zero.
  - The characteristic exponent of the Lévy process is $\psi_X(u) = \frac{1}{2} u^2 \sigma^2 + \lambda \left(1 - e^{iu\mu_J - \frac{1}{2} u^2 v_J}\right)$. The cumulant exponent is $\kappa_X(s) = \frac{1}{2} s^2 \sigma^2 + \lambda \left(e^{s\mu_J + \frac{1}{2} s^2 v_J} - 1\right)$.

- We can write the return as $\ln S_t/S_0 = \mu t + X_t - \left(\frac{1}{2} \sigma^2 + \lambda \left(e^{\mu_J + \frac{1}{2} v_J} - 1\right)\right) t$.

- The characteristic function of the return is:
  $\phi(u) = e^{iu\mu t} e^{-\frac{1}{2} \sigma^2 (u^2+iu)t} e^{-\lambda \left(1-e^{iu\mu_J - \frac{1}{2} u^2 v_J}\right) + iu\lambda \left(e^{\mu_J + \frac{1}{2} v_J} - 1\right)} t$.

- Under $Q$, $\mu = r - q$. 

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Dampened power law (DPL)

- The driver of this model is a pure jump Lévy process, with its characteristic exponent

\[ \psi_X(u) = -\lambda \Gamma(-\alpha) \left[ (\beta_+ - iu)^\alpha - \beta_+^\alpha + (\beta_- + iu)^\alpha - \beta_-^\alpha \right] - iuC(h). \]

The cumulant exponent is

\[ \kappa_X(s) = \lambda \Gamma(-\alpha) \left[ (\beta_+ - s)^\alpha - \beta_+^\alpha + (\beta_- + s)^\alpha - \beta_-^\alpha \right] + sC(h). \]

- We can write the return as, \( \ln S_t/S_0 = \mu t + X_t - \kappa_X(1)t. \)

- The characteristic function of the return is:

\[ \phi(u) = e^{iu\mu t} e^{-\lambda \Gamma(-\alpha) \left[ (\beta_+ - iu)^\alpha - \beta_+^\alpha + (\beta_- + iu)^\alpha - \beta_-^\alpha \right] + iu\lambda \Gamma(-\alpha) \left[ (\beta_+ - 1)^\alpha - \beta_+^\alpha + (\beta_- + 1)^\alpha - \beta_-^\alpha \right]} t. \]

- Under \( \mathbb{Q} \), \( \mu = r - q. \)

- The characteristic exponent of the convexity adjusted Lévy process \( (X_t - \kappa_X(1)t) \) is: \( \psi_X(u) + iu\kappa_X(1). \)

References:


Special cases of DPL

- **α-stable law**: No exponential dampening, $\beta_{\pm} = 0$.
  Without exponential dampening, return moments greater than $\alpha$ are no longer well defined.
  Characteristic function takes different form to account singularity.

- **Variance gamma (VG) model**: $\alpha = 0$.
  The characteristic exponent takes a different form as $\alpha = 0$ represents a singular point ($\Gamma(0)$ not well defined).

- **Double exponential model**: $\alpha = -1$.
Other Lévy examples

- Other examples:
  - The normal inverse Gaussian (NIG) process of Barndorff-Nielsen (1998)
  - The generalized hyperbolic process (Eberlein, Keller, Prause (1998))
  - The Meixner process (Schoutens (2003))
  - ...

- Bottom line:
  - All tractable in terms of analytical characteristic exponents $\psi(u)$.
  - We can use FFT to generate the density function of the innovation (for model estimation).
  - We can also use FFT to compute option values.
Why normal?

- Traditional asset pricing theories all invariably start with a normal distribution (or a Brownian motion in continuous time).

  
  *The normal law of error stands out in the experience of mankind as one of the broadest generalizations of natural philosophy. It serves as the guiding instrument in researches in the physical and social sciences and in medicine, agriculture and engineering. It is an indispensable tool for the analysis and the interpretation of the basic data obtained by observation and experiment.*

- G. Lippman, quoted in D’Arcy Thompson’s *On Growth and Form* V. I, p. 121:
  
  *Everybody believes in the normal approximation, the experimenters because they believe it is a mathematical theorem, the mathematicians because they believe it is an experimental fact!*
Why normal?


  to quote a statement of Poincaré, who said (partly in jest no doubt) that there must be something mysterious about the normal law, since mathematicians think it is a law of nature whereas physicists are convinced that it is a mathematical theorem.

- Dr. O. Lord, Language expert and scholar:

  Generally Assumed Ubiquitous Symmetric Shape Identifying Additive Noise

- Following the well-known law that every named quantity in mathematics was invented by somebody else, the credit for discovering the Gaussian bell-shaped curve should actually go to Abraham De Moivre, who discovered it in 1733.

- Gauss and Laplace rediscovered it in 1809 and 1812 respectively in their work on the theory of errors in observation.
The delusion of diffusion

- Starting with Bachelier (1900), diffusion processes have been the most widely used class of stochastic processes used to describe the evolution of asset prices over time.

- The sample paths of diffusion processes are continuous over time, but are nowhere differentiable. The mathematical model is the idealization of the trajectory of a single particle being constantly bombarded by an infinite number of infinitesimally small random forces.

- Like a shark, a diffusion process must always be moving, or else it dies.

- As Woody Allen said to Diane Keaton in describing their relationship, “I think what we have on our hands here is a dead shark” (Annie Hall).
Infinite variation of diffusion sample paths

- In even the most active markets, one can find small enough time periods over which there is positive probability of no price change.
- Furthermore, if we sum the absolute values of price changes over a day, we get a finite number.
- Diffusion processes have neither property. Over any finite time interval, there is zero probability that the price does not change. Furthermore, the absolute values of price changes over a day (or any other period) sum to infinity.
- If one tried to accurately draw a diffusion sample path, your pen would run out of ink before one second had elapsed.
- The failure of diffusions to describe the microscopic behavior of sample paths would not be troubling if financial theory took a more macroscopic view.
- The problem is that the foundations of standard financial theories such as Black Scholes and the intertemporal CAPM rest on the ability of investors to continuously rebalance their portfolios.
- Nobody seriously believes that anyone can trade continuously and even if they could, no one seriously believes that trade sizes can be kept so small that price impact is infinitesimal.
Time for a change

- If we confront a mathematical diffusion with real-life sample paths, the modelling question becomes one of finding ways to slow down diffusions in order to more accurately capture dynamics.

- In 1949, Bochner introduced the notion of time change to stochastic processes. In 1973, Clark suggested that time-changed diffusions could be used to accurately describe financial time series.

- Mathematically, a clock is just a weakly increasing stochastic process started at zero. When one time changes a stochastic process, this clock is used to index a stochastic process such as a diffusion. Clark suggested that the price process runs on business time, while business time itself increases weakly over calendar time.

- The possibility that business time may not move while calendar time inexorably marches forward is important for our purposes.
Classifying clocks

- At present, there are two types of clocks used to model business time:
  1. Continuous clocks have the property that business time is always strictly increasing over calendar time.
  2. Clocks based on increasing jump processes have staircase like paths.
- The first type of clock can be used to describe stochastic volatility models — next chapter.
- The second type of clock has the capability of slowing a diffusion process down to market speeds.
  - A subordinator is a pure jump increasing process with stationary independent increments.
Run Brownian motions on different subordinators as business clocks

- If the clock is a **standard Poisson process**, with unit jump sizes and iid exponentially distributed inter-jump times:
  ⇒ The resulting process is a **compound Poisson process** with normal jump sizes.

- If the clock is a **compound Poisson process** with exponentially distributed jump size with mean one:
  ⇒ **DPL with** $\alpha = -1$ Compound Poisson with Laplace (two-sided exponential) jump sizes.
  ⇒ Asymmetry can be induced by running the clock on a diffusion with drift $(\mu t + W_t)$ instead of a standard Brownian motion.

- If the clock is a **gamma process**
  ⇒ **DPL with** $\alpha = 0$ (variance gamma). Asymmetry can be induced by running the clock on a diffusion with drift $(\mu t + W_t)$ instead of a standard Brownian motion.
  **continuous process.**
General evidence on Lévy return innovations

- **Credit risk:** (compound) Poisson process
  - The whole intensity-based credit modeling literature...

- **Market risk:** Infinite-activity jumps
  - Evidence from stock returns (CGMY (2002)): The $\alpha$ estimates for DPL on most stock return series are greater than zero.
  - Evidence from options: Models with infinite-activity return innovations price equity index options better (Carr & Wu (2003), Huang & Wu (2004))

- The role of diffusion (in the presence of infinite-variation jumps)
  - Not big, difficult to identify (CGMY (2002), Carr & Wu (2003a,b)).
  - Generate correlations with diffusive activity rates (Huang & Wu (2004)).
  - The diffusion ($\sigma^2$) is identifiable in theory even in presence of infinite-variation jumps (Aït-Sahalia (2004), Aït-Sahalia & Jacod 2005).
Implied volatility smiles & skews on a stock

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Implied volatility skews on a stock index (SPX)

SPX: 17–Jan–2006

More skews than smiles

Maturities: 32, 60, 151, 242, 333, 704

Implied Volatility

Moneyness = $\ln\left(\frac{K}{F}\right) \cdot \frac{\sigma}{\sqrt{\tau}}$
Average implied volatility smiles on currencies

Maturities: 1m (solid), 3m (dashed), 1y (dash-dotted)
The role of jumps at very short maturities

- Implied volatility smiles (skews) ↔ non-normality (asymmetry) for the risk-neutral return distribution.
  
  \[ IV(d) \approx ATMV \left( 1 + \frac{\text{Skew.}}{6} d + \frac{\text{Kurt.}}{24} d^2 \right), \quad d = \frac{\ln K/F}{\sigma \sqrt{\tau}} \]

- Two mechanisms to generate return non-normality:
  - Use Lévy jumps to generate non-normality for the innovation distribution.
  - Use stochastic volatility to generates non-normality through mixing over multiple periods.

- Over very short maturities (1 period), *only jumps contribute to return non-normalities*. 

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Time decay of short-term OTM options


- As option maturity ↓ zero, OTM option value ↓ zero.
- The speed of decay is exponential $O(e^{-c/T})$ under pure diffusion, but linear $O(T)$ in the presence of jumps.
- Term decay plot: $\ln(OTM/T) \sim \ln(T)$ at fixed $K$:

In the presence of jumps, the Black-Scholes implied volatility for OTM options $IV(\tau, K)$ explodes as $\tau \downarrow 0$. 

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The impacts of jumps at very long horizons

- Central limit theorem (CLT): Return distribution converge to normal with aggregation under certain conditions (finite return variance,...)
  ⇒ As option maturity increases, the smile should flatten.
- Evidence: The skew does not flatten, but steepens!
  - Return variance is infinite. ⇒ CLT does not apply.
  - Down jumps only. ⇒ Option has finite value.
- But CLT seems to hold fine statistically:
Reconcile $\mathbb{P}$ with $\mathbb{Q}$ via DPL jumps


- Model return innovations under $\mathbb{P}$ by DPL:

$$
\pi(x) = \begin{cases} 
\lambda \exp(-\beta_+ x) x^{-\alpha-1}, & x > 0, \\
\lambda \exp(-\beta_- |x|) |x|^{-\alpha-1}, & x < 0.
\end{cases}
$$

All return moments are finite with $\beta_+ > 0$. *CLT applies.*

- Market price of jump risk ($\gamma$):  
  $$
  \left. \frac{d\mathbb{Q}}{d\mathbb{P}} \right|_t = \mathbb{E}(-\gamma X)
  $$

- The return innovation process remains DPL under $\mathbb{Q}$:

$$
\pi(x) = \begin{cases} 
\lambda \exp(-(\beta_+ + \gamma) x) x^{-\alpha-1}, & x > 0, \\
\lambda \exp(-(\beta_- - \gamma) |x|) |x|^{-\alpha-1}, & x < 0.
\end{cases}
$$

- To break CLT under $\mathbb{Q}$, set $\gamma = \beta_-$ so that $\beta_\mathbb{Q} = 0$.

- Reconciling $\mathbb{P}$ with $\mathbb{Q}$: *Investors charge maximum allowed market price on down jumps.*
(III) Default risk & long-term implied vol skew

- When a company defaults, its stock value jumps to zero.
- It generates a steep skew in long-term stock options.
- Evidence: Stock option implied volatility skews are correlated with credit default swap (CDS) spreads written on the same company.


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Three Lévy jump components in stock returns

I. Market risk (FMLS under $\mathcal{Q}$, DPL under $\mathbb{P}$)
   ▶ The stock index skew is strongly negative at long maturities.

II. Idiosyncratic risk (DPL under both $\mathbb{P}$ and $\mathcal{Q}$)
   ▶ The smile on single name stocks is not as negatively skewed as that on index at short maturities.

III. Default risk (Compound Poisson jumps).
   ▶ Long-term skew moves together with CDS spreads.

Information and identification:
   ▶ Identify market risk from stock index options.
   ▶ Identify the credit risk component from the CDS market.
   ▶ Identify the idiosyncratic risk from the single-name stock options.
Lévy jump components in currency returns

- Model currency return as the difference of the log pricing kernels between the two economies.
- Pricing kernel assigns market prices to systematic risks.
- Market risk dominates for exchange rates between two industrialized economies (e.g., dollar-euro).
  - Use a one-sided DPL for each economy (downward jump only).
- Default risk shows up in FX for low-rating economies (say, dollar-peso).
  - Peso drops by a large amount when the country (Mexico) defaults on its foreign debt.


- When pricing options on exchange rates, it is appropriate to distinguish between world risk versus country-specific risk.

Pricing kernel: Review

- In the absence of arbitrage, there exists at least one strictly positive process, $M_t$, the state-price deflator, in each economy such that the deflated gains process associated with any admissible strategy is a martingale. The ratio of $M_t$ at two horizons, $M_{t,T}$, is called a stochastic discount factor, or more informally, a pricing kernel.

- For an asset that has a terminal payoff $\Pi_T$, its time-$t$ value is $p_t = \mathbb{E}_t^P [M_{t,T}\Pi_T]$.

- In a discrete-time representative agent economy with an additive CRRA utility, the pricing kernel is equal to the ratio of the marginal utilities of consumption,

$$M_{t,t+1} = \beta \frac{u'(c_{t+1})}{u'(c_t)} = \beta \left( \frac{c_{t+1}}{c_t} \right)^{-\gamma}$$

- In an exchange economy, $c_t = w_t$, $M_{t,t+1} = \beta e^{-\gamma \ln w_{t+1}/w_t}$, where the log return on aggregate wealth $\ln w_{t+1}/w_t$ can be approximated by log return on a stock index/market portfolio.
The multiplicative representation of the pricing kernel

\[ M_{t,T} = \exp \left( - \int_t^T r_s \, ds \right) \mathcal{E} \left( - \int_t^T \gamma_s^\top dX_s \right) \]

\( r \) is the instantaneous riskfree rate, \( \mathcal{E} \) is the stochastic exponential martingale operator,

- \( X \) denotes the risk sources in the economy. In the exchange economy, \( X \) denotes the log return on the aggregate wealth, \( X_t = \ln w_t/w_0 \).
- \( \gamma \) is the market price of the risk \( X \). If \( X \) is return on the aggregate wealth, \( \gamma \) would be the relative risk aversion.
- In case of Brownian risk, constant volatility, constant risk aversion, and constant interest rate, we have \( M_{t,T} = e^{-r\tau} e^{\left( -\gamma \sigma (W_T - W_t) - \frac{1}{2} \gamma^2 \sigma^2 \tau \right)} \).
- In case of Lévy risk, constant risk aversion, and constant interest rate, we have \( M_{t,T} = e^{-r\tau} e^{\left( -\gamma (X_T - X_t) - \kappa_X(-\gamma) \tau \right)} \).
From pricing kernels to exchange rates

- Let $S_{t}^{fh}$ denote the time-$t$ currency-$h$ price of currency $f$, with $h$ being home and $f$ denoting foreign.

- Currency returns are related to the pricing kernels of the two economies by
  \[
  \ln S_{T}^{fh}/S_{t}^{fh} = \ln M_{t,T}^{f} - \ln M_{t,T}^{h}
  \]
  Example, $S$ is dollar price of pound ($1.9$ per pound), $f$ would be UK, and $h$ would be US.

- When each economy’s risk is modeled by a Lévy process with constant relative risk aversion and constant interest rates,
  \[
  M_{t,T}^{i} = e^{-r^{i} \tau} e^{(-\gamma^{i} (X_{T}^{i} - X_{t}^{i}) - \kappa X^{i} (-\gamma^{i}) \tau)} \text{ with } i = h, f,
  \]
  we have
  \[
  \ln S_{T}^{fh}/S_{t}^{fh} = (r^{d} - r^{f}) \tau + \gamma^{h} (X_{T}^{h} - X_{t}^{h}) + \kappa X^{h} (-\gamma^{h}) \tau - \gamma^{f} (X_{T}^{f} - X_{t}^{f}) - \kappa X^{f} (-\gamma^{f}) \tau
  \]

- **Under the above Lévy specification, what’s the expected excess return (risk premium) on the currency investment?** (assume independence between $X^{h}$ and $X^{f}$ for simplicity)

  Answer: $\kappa X^{h} (\gamma^{h}) + \kappa X^{h} (-\gamma^{h})$. 
Beyond Lévy processes

- Lévy processes can be used to generate different iid return innovation distributions.

- Yet, return distribution is not iid. It varies stochastically over time.

- We need to go beyond Lévy processes to capture the stochastic nature of the return distribution.

- Applying separate stochastic time changes to different Lévy components generates
  - separate stochastic responses to each economic shock.
  - stochastic volatility, skewness, ...
Economic implications of using jumps

- In the Black-Scholes world (one-factor diffusion):
  - The market is complete with a bond and a stock.
  - The world is risk free after delta hedging.
  - Utility-free option pricing. Options are redundant.

- In a pure-diffusion world with stochastic volatility:
  - Market is complete with one (or a few) extra option(s).
  - The world is risk free after delta and vega hedging.

- In a world with jumps of random sizes:
  - The market is inherently incomplete (with stocks alone).
  - Need all options (+ model) to complete the market.
  - Derman: “Beware of economists with Greek symbols!”
  - Options market is informative/useful:
    - Cross sections \((K, T) \leftrightarrow \mathbb{Q}\) dynamics.
    - Time series \((t) \leftrightarrow \mathbb{P}\) dynamics.
    - The difference \(\mathbb{Q}/\mathbb{P}\) \(\leftrightarrow\) market prices of economic risks.
Different types of jumps affect option pricing at both short and long maturities.

- Implied volatility smiles at very short maturities can only be accommodated by a jump component.
- Implied volatility skews at very long maturities ask for a jump process that generates infinite variance.
- Credit risk exposure may also help explain the long-term skew on single name stock options.

The choice of jump types depends on the events:

- Infinite-activity jumps $\Leftrightarrow$ frequent market order arrival.
- Finite-activity Poisson jumps $\Leftrightarrow$ rare events (credit).

The presence of jumps of random sizes creates value for the options markets...