Options: Definitions, Payoffs, & Replications

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Options Markets
Definitions and terminologies

- An option gives the option holder the right/option, but no obligation, to buy or sell a security to the option writer/seller
  - for a pre-specified price (the strike price, $K$)
  - at (or up to) a given time in the future (the expiry date)

- An option has positive value.
  *Comparison: a forward contract has zero value at inception.*

- Option types
  - A call option gives the holder the right to buy a security. The payoff is $(S_T - K)^+$ when exercised at maturity.
  - A put option gives the holder the right to sell a security. The payoff is $(K - S_T)^+$ when exercised at maturity.
  - American options can be exercised at any time prior to expiry.
  - European options can only be exercised at the expiry.
More terminologies

- **Moneyness**: the strike relative to the spot/forward level
  - An option is said to be **in-the-money** if the option has positive value if exercised right now:
    - $S_t > K$ for call options and $S_t < K$ for put options.
    - Sometimes it is also defined in terms of the forward price at the same maturity (in the money forward): $F_t > K$ for call and $F_t < K$ for put.
    - The option has positive **intrinsic value** when in the money. The intrinsic value is $(S_t - K)^+$ for call, $(K - S_t)^+$ for put.
    - We can also define intrinsic value in terms of forward price.
  - An option is said to be **out-of-the-money** when it has zero intrinsic value.
    - $S_t < K$ for call options and $S_t > K$ for put options.
    - **Out-of-the-money forward**: $F_t < K$ for call and $F_t > K$ for put.
  - An option is said to be **at-the-money** spot (or forward) when the strike is equal to the spot (or forward).
More terminologies

- The value of an option is determined by:
  - the current spot (or forward) price \( S_t \) or \( F_t \),
  - the strike price \( K \),
  - the time to maturity \( \tau = T - t \),
  - the option type (Call or put, American or European), and
  - the dynamics of the underlying security (e.g., how volatile the security price is).

- Out-of-the-money options do not have intrinsic value, but they have time value.

- Time value is determined by time to maturity of the option and the dynamics of the underlying security.

- Generically, we can decompose the value of each option into two components:
  \[
  \text{option value} = \text{intrinsic value} + \text{time value}.
  \]
Assets underlying exchanged-traded options

- Stocks (OMON)
- Stock indices
- Index return variance (new)
- Exchange rate (XOPT)
- Futures
Specification of exchange-traded options

- Expiration date \((T)\)
- Strike price \((K)\)
- European or American
- Call or Put (option class)

OTC options (such as OTC options on currencies) are quoted differently.
Options market making

- Most exchanges use **market makers** to facilitate options trading.
- A market maker is required to provide bid and ask quotes
  - with the bid-ask spread within a maximum limit,
  - with the size no less than a minimum requirement,
  - at no less than a certain percentage of time (lower limit)
  - on no less than a certain fraction of securities that they cover.
- The benefit of market making is the bid-ask spread; The risk is market movements.
  - The risk and cost of options market making is relatively large.
  - The bid-ask is wide (stock options). The tick size is 10 cents on options with prices higher than $3. It is 5 cents otherwise.
Options market making

- Since there can be hundreds of options underlying one stock, when the stock price moves, quotes on the hundreds of options must be updated simultaneously.
  - Quote message volume is dramatically larger than trade message volume.
  - The risk exposure is large compared to the benefit.
    - When a customer who has private information on the underlying stock (say, going up), the customer can buy all the call options and sell all the put options underlying one stock.
    - The market maker’s risk exposure is the sum of all the quote sizes he honors on each contract.
    - Market makers hedge their risk exposures by buying/selling stocks according to their option inventories.
  - Market makers nowadays all have automated systems to update their quotes, and calculate their optimal hedging ratios.
  - Options market makers are no longer individual persons, but are well-capitalized firms.
Exchanges that trade stock options in the US

A rapidly changing landscape

- Old exchanges:
  - CBOE (Chicago Board of Options Exchange)
  - AMEX (American Stock Exchange)
  - PCX (Pacific Stock Exchange)
  - PHLX (Philadelphia Stock Exchange)

- New developments:
  - BOX (Boston Options Exchange, February 2004)
Margins

- Margins are required when options are sold/written.

- When a naked option is written the margin is the greater of:
  
  - A total of 100% of the proceeds of the sale plus 20% of the underlying share price less the amount (if any) by which the option is out of the money
  
  - A total of 100% of the proceeds of the sale plus 10% of the underlying share price.

- For other trading strategies there are special rules
For European options, the terminal payoff can be written as \((S_T - K)^+\) for calls and \((K - S_T)^+\) for puts at expiry date \(T\).

Since options have positive value, one needs to pay an upfront price (option price) to possess an option.

The P&L from the option investment is the difference between the terminal payoff and the initial price you pay to obtain the option.

Do not confuse the two.

The textbook likes to talk about P&Ls, but I like to talk about payoffs — Different perspectives:

- P&Ls: If I buy/sell an option today, how much money I can make under different scenarios? What’s my return?
- Payoffs: If I desire a certain payoff structure in the future, what types of options/positions I need to generate it?
An example: Call option on a stock index

Consider a European call option on a stock index. The current index level (spot $S_t$) is 100. The option has a strike ($K$) of $90$ and a time to maturity ($T - t$) of 1 year. The option has a current value ($c_t$) of $14$.

- Is this option in-the-money or out-of-the-money (wrt to spot)?
- What’s intrinsic value for this option? What’s its time value?
- If you hold this option, what’s your terminal payoff?
  - What’s your payoff and P&L if the index level reaches 100, 90, or 80 at the expiry date $T$?
- If you write this option and have sold it to the exchange, what does your terminal payoff look like?
  - What’s your payoff and P&L if the index level reaches 100, 90, or 80 at the expiry date $T$?
Payoffs and P&Ls from long/short a call option

\[(S_t = 100, K = 90, c_t = 14)\]

**Long a call pays off,** \((S_T - K)^+\), **bets on index price going up.**

**Shorting a call bets on index price going down.**
Another example: Put option on an exchange rate

Consider a European put option on the dollar price of pound (GBPUSD). The current spot exchange rate \( S_t \) is $1.6285 per pound. The option has a strike \( K \) of $1.61 and a time to maturity \( T - t \) of 1 year. The 1-year forward price \( F_{t,T} \) is $1.61. The dollar continuously compounding interest rate at 1-year maturity \( r_d \) is 5\%. The option \( p_t \) is priced at $0.0489.

- From the above information, can you infer the continuously compounding interest rate at 1-year maturity on pound \( r_f \)?

- Is this option in-the-money or out-of-the-money wrt to spot? What’s the moneyness in terms of forward?

- In terms of forward, what’s intrinsic value for this option? What’s its time value?

- If you hold this option, what’s your terminal payoff, if the dollar price of pound reaches 1.41, 1.61, or 1.81 at the expiry date \( T \)?
Another example: Put option on an exchange rate

- Review the forward pricing formula: $F_{t,T} = S_t e^{(r_d-r_f)(T-t)}$.
  
  $r_f = r_d - \frac{1}{T-t} \ln(F_{t,T}/S_t) = 0.05 - \ln(1.61/1.6285)/1 = 6.14\%$.

  - Recall covered interest rate parity: Annualized forward return $\left( \frac{1}{T-t} \ln(F_{t,T}/S_t) \right)$ on exchange rates equals interest rate differential $(r_d - r_f)$ between the two currencies.

- Long a put option pays off, $(K - S_T)^+$, and bets on the underlying currency (pound) depreciates.

- Shorting a put option bets on pound appreciates.

- *How does it differ from betting using forwards?*
Payoffs and P&Ls from long/short a put option

\((S_t = 1.6285, F_{t,T} = 1.61, K = 1.61, p_t = 0.0489)\)
What derivative positions generate the following payoff?
Put-call conversions

Plot the payoff function of the following combinations of calls/puts and forwards at the same strike $K$ and maturity $T$.

1. Long a call, short a forward.
   ▶ Compare the payoff to long a put.

2. Short a call, long a forward.
   ▶ Compare the payoff to short a put.

3. Long a put, long a forward.
   ▶ Compare the payoff to long a call.

4. Short a put, short a forward.
   ▶ Compare the payoff to short a call.

5. Long a call, short a put.
   ▶ Compare the payoff to long a forward.

6. Short a call, long a put.
   ▶ Compare the payoff to short a forward.
Put-call conversions: Payoff comparison ($K = 100$)

The dash and dotted lines are payoffs for the two composition instruments. The solid lines are payoffs of the target.
The linkage between put, call, and forward

- The above conversions reveal the following parity condition in payoffs of put, call, and forward at the same strike and maturity:
  
  Payoff from a call − Payoff from a forward = Payoff from a put
  Payoff from a put + Payoff from a forward = Payoff from a call
  Payoff from a call − Payoff from a put = Payoff from a forward

- If the payoff is the same, the present value should be the same — put-call parity:
  
  \[ c_t - p_t = e^{-r(T-t)}(F_{t,T} - K). \]

- At a fixed strike \((K)\) and maturity \(T\), we only need to know the two prices of the following three: \((c_t, p_t, F_{t,T})\).
  One of the three contracts is redundant.
Intrinsic values and time values, Revisited

- A more formal definition:

\[ c_t = e^{-r(T-t)}(F_{t,T} - K)^+ + TV_t, \quad p_t = e^{-r(T-t)}(K - F_{t,T})^+ + TV_t. \]

- Call and put at the same maturity & strike \((T, K)\) have the same time.

- Most option pricing models only deal with the time value, and take the forward (financing) and hence intrinsic value as given (starting point).

- To estimate these option pricing models, one only needs to use the time value at each \((T, K)\), which is the out-of-the-money option (call when \(F_t < K\) and put otherwise).

  - Normally, the out-of-the-money option is more actively traded, possibly for the same reason.
    — Most professional options traders are trading the time value only.
In the absence of forward, use spot and bond:

- Can you use a spot and bond to replicate a forward payoff?
- What’s the payoff function of a zero bond?
Popular payoff I: Bull spread

Can you generate the above payoff structure (solid blue line) using (in addition to cash/bond):

- two calls
- two puts
- a call, a put, and a stock/forward

Who wants this type of payoff structure?
Generating a bull spread

- **Two calls**: Long call at $K_1 = $90, short call at $K_2 = $110, short a bond with $10 par.

- **Two puts**: Long a put at $K_1 = $90, short put at $K_2 = $110, long a bond with $10 par.

- **A call, a put, and a stock/forward**: Long a put at $K_1 = $90, short a call at $K_2 = $110, long a forward at $K = 100$ (or long a stock, short a bond at $100$ par).
Pointers in replicating payoffs

- Each kinky point corresponds to a strike price of an option contract.
- Given put-call party, you can use either a call or a put at each strike point.
- Use bonds for parallel shifts.
- A general procedure using calls, forwards, and bonds
  - Starting from the left side of the payoff graph at $S_T = 0$ and progress to each kinky point sequentially to the right.
  - If the payoff at $S_T = 0$ is $x$ dollars, long a zero-coupon bond with an $x$-dollar par value. [Short if $x$ is negative].
  - If the slope of the payoff at $S_T = 0$ is $s_0$, long $s_0$ shares of a call/forward with a zero strike — A call at zero strike is the same as a forward at zero strike. [Short if $s_0$ is negative.]
  - Go to the next kinky point $K_1$. If the next slope (to the right of $K_1$ is $s_1$, long $(s_1 - S_0)$ shares of call at strike $K_1$. Short when the slope change is negative.
  - Go to the next kinky point $K_2$ with a new slope $s_2$, and long $(s_2 - s_1)$ shares of calls at strike $K_2$. Short when the slope change is negative.
  - Keep going until there are no more slope changes.
Pointers in replicating payoffs, continued

- A general procedure using puts, forwards, and bonds
  - Starting from the right side of the payoff graph at the highest strike under which there is a slope change. Let this strike be $K_1$.
  - If the payoff at $K_1$ is $x$ dollars, long a zero-coupon bond with an $x$-dollar par value. [Short if $x$ is negative].
  - If the slope to the right of $K_1$ is positive at $s_0$, long $s_0$ of a forward at $K_1$. Short the forward if $s_0$ is negative.
  - If the slope to the left of $K_1$ is $s_1$, short $(s_1 - s_0)$ shares of a put at $K_1$. Long if $(s_1 - s_0)$ is negative.
  - Go to the next kinky point $K_2$. If the slope to the left of $K_2$ is $s_2$, short $(s_2 - s_1)$ put with strike $K_2$.
  - Keep going until there are no more slope changes.
Example: Bear spread

- How many (at minimum) options do you need to replicate the bear spread?
- Do the exercise, get familiar with the replication.
- Who wants a bear spread?
Example: Straddle

- How many (at minimum) options do you need to replicate the straddle?
- Do the exercise, get familiar with the replication.
- Who wants long/short a straddle?
Example: Strangle

- How many (at minimum) options do you need to replicate the strangle?
- Do the exercise, get familiar with the replication.
- Who wants long/short a strangle?
How many (at minimum) options do you need to replicate the butterfly spread?

Do the exercise, get familiar with the replication.

Who wants long/short a butterfly spread?
Example: Risk Reversal

- How many (at minimum) options do you need to replicate the risk reversal?
- Do the exercise, get familiar with the replication.
- Who wants long/short a risk reversal?
Smooth out the kinks: Can you replicate this?

How many options do you need to replicate this quadratic payoff?
- You need a continuum of options to replicate this payoff.
- The weight on each strike $K$ is $2dK$.

Who wants long/short this payoff?
- The variance of the stock price is $\mathbb{E}[(S_T - F_{t,T})^2]$.
- Variance swap contracts on major stock indexes are actively traded.
Replicate any terminal payoff with options and forwards

\[ f(S_T) = f(F_t) + f'(F_t)(S_T - F_t) \]

\[ + \left\{ \int_0^{F_t} f''(K)(K - S_T)^+ dK + \int_{F_t}^{\infty} f''(K)(S_T - K)^+ dK \right\} \]

\[ \text{bonds} \quad \text{forwards} \quad \text{OTM options} \]

- What does this formula tell you?
  - With bonds, forwards, and European options, we can replicate any terminal payoff structures.
  - More exotic options deal with path dependence, correlations, etc.

- You do not need to memorize the formula.
- Can you reconcile this formula with my tips on using calls alone, or puts alone to replicate any payoffs?
- Can you provide some tips (suggested steps) to replicate any payoffs using out-of-the-money options only (plus cash & forwards)?
- Code them up and see if they work.
Replication applications

- Replicate the return variance swap using options and futures.

- Based on the replication idea, think of ways to summarizing the information in the options market.
  - Information about the directional movement of the underlying.
  - Information about return variance.
  - Information about large movements of a certain direction.
  - Information about large movements of either direction.

- Examples:
  - Jianfeng Hu.
  - Put-call parity violations and stock returns

- Caveat: Far out-of-the-money options may not be actively traded. Quotes may not be reliable.
  Example: ATM volatility versus synthetic variance swap.