Using Lévy Processes to Model Return Innovations

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Option Pricing
Outline

1. Lévy processes
2. Lévy characteristics
3. Examples
4. Evidence
5. Jump design
6. Economic implications
A Lévy process is a continuous-time process that generates stationary, independent increments ...

Think of return innovations ($\varepsilon$) in discrete time: $R_{t+1} = \mu_t + \sigma_t \varepsilon_{t+1}$.

- Normal return innovation — diffusion
- Non-normal return innovation — jumps

Classic Lévy specifications in finance:
- Brownian motion (Black-Scholes, Merton)
- Compound Poisson process with normal jump size (Merton)

$\Rightarrow$ The return innovation distribution is either normal or mixture of normals.
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Lévy characteristics

- Lévy processes greatly expand our continuous-time choices of iid return innovation distributions via the Lévy triplet \((\mu, \sigma, \pi(x))\). \((\pi(x)\)–Lévy density).

- The Lévy-Khintchine Theorem:
  \[
  \phi_{X_t}(u) \equiv \mathbb{E}[e^{iuX_t}] = e^{-t\psi(u)},
  \]
  \[
  \psi(u) = -iu\mu + \frac{1}{2}u^2\sigma^2 + \int_{\mathbb{R}_0} (1 - e^{iu} + iux1_{|x|<1}) \pi(x)dx,
  \]

- Innovation distribution
  \(\leftrightarrow\) characteristic exponent \(\psi(u)\)
  \(\leftrightarrow\) Lévy triplet \((\mu, \sigma, \pi(x))\)
  - Constraint: \(\int_{\mathbb{R}_0} x^21_{|x|<1}\pi(x)dx < \infty\).
  - “Tractable:” if the integral can be carried out explicitly.

- When well-defined, it is convenient to define the cumulant exponent:
  \[
  \kappa(s) \equiv \frac{1}{t} \ln \mathbb{E}[e^{sX_t}] = s\mu + \frac{1}{2}s^2\sigma^2 + \int_{\mathbb{R}_0} (e^{sx} - 1 - sx1_{|x|<1}) \pi(x)dx.
  \]
  \[
  \psi(u) = -\kappa(iu), \quad \kappa(s) = -\psi(-is).
  \]
Model stock returns with Lévy processes

- Let $X_t$ be a Lévy process, $\kappa_X(s)$ its cumulant exponent
- The log return on a security can be modeled as
  \[
  \ln \frac{S_t}{S_0} = \mu t + X_t - t\kappa_X(1)
  \]
  where $\mu$ is the instantaneous drift (mean) of the stock such that
  \[
  \mathbb{E}[S_t] = S_0 e^{\mu t}.
  \]
  The last term $-t\kappa_X(1)$ is a convexity adjustment such that $X_t - t\kappa_X(1)$ forms an exponential martingale:
  \[
  \mathbb{E}\left[e^{X_t - t\kappa_X(1)}\right] = 1.
  \]
- Since both $\mu$ and $\kappa_X(1)$ are deterministic components, they can be combined together: $\ln \frac{S_t}{S_0} = mt + X_t$, but it is more convenient to separate them so that the mean instantaneous return $\mu$ is kept as a separate free parameter.
- Under $\mathbb{Q}$, $\mu = r - q$.
- Under this specification, we shall always set the first component of the Lévy triplet to zero $(0, \sigma, \pi(x))$, because it will be canceled out with the convexity adjustment.
Characteristic function of the security return

\[ s_t \equiv \ln S_t / S_0 = \mu t + X_t - t\kappa X(1) \]

- The characteristic function for the security return is

\[ \phi_{s_t}(u) \equiv \mathbb{E} \left[ e^{iu \ln S_t / S_0} \right] = \exp \left( - [-i\mu u + \psi_X(u) + iu\kappa X(1)] t \right) \]

- The characteristic exponent is

\[ \psi_{s_t}(u) = -i\mu u + \psi_X(u) + iu\kappa X(1) \]

- Under \( \mathbb{Q} \), \( \mu = r - q \). The focus of the model specification is on \( X_t \sim (0, \sigma, \pi(x)) \), unless \( r \) and/or \( q \) are modeled to be stochastic.
Tractable examples of Lévy processes

1. Brownian motion (BSM) \((\mu t + \sigma W_t)\): normal shocks.
2. Compound Poisson jumps (Merton, 76): Large but rare events.
   \[
   \pi(x) = \lambda \frac{1}{\sqrt{2\pi \nu_J}} \exp \left( - \frac{(x - \mu_J)^2}{2\nu_J} \right).
   \]
3. Dampened power law (DPL):
   \[
   \pi(x) = \begin{cases} 
   \lambda \exp(-\beta_+ x) x^{-\alpha-1}, & x > 0, \\
   \lambda \exp(-\beta_- |x|) |x|^{-\alpha-1}, & x < 0,
   \end{cases}
   \]
   \[\lambda, \beta_\pm > 0, \quad \alpha \in [-1, 2)\]

- **Finite activity** when \(\alpha < 0\): \(\int_{\mathbb{R}^0} \pi(x) dx < \infty\). Compound Poisson. Large and rare events.
- **Infinite activity** when \(\alpha \geq 0\): Both small and large jumps. Jump frequency increases with declining jump size, and approaches infinity as \(x \to 0\).
- **Infinite variation** when \(\alpha \geq 1\): many small jumps.

*Market movements of all magnitudes, from small movements to market crashes.*
Analytical characteristic exponents

- **Diffusion:** \( \psi(u) = -i\mu + \frac{1}{2}u^2\sigma^2. \)

- **Merton’s compound Poisson jumps:**
  \[ \psi(u) = \lambda \left( 1 - e^{i\mu - \frac{1}{2}u^2\nu} \right). \]

- **Dampened power law:** (for \( \alpha \neq 0, 1 \))
  \[ \psi(u) = -\lambda \Gamma(-\alpha) \left[ (\beta_+ - iu)^\alpha - \beta_+^\alpha + (\beta_- + iu)^\alpha - \beta_-^\alpha \right] - iuC(h). \]
  - When \( \alpha \to 2 \), smooth transition to diffusion (quadratic function of \( u \)).
  - When \( \alpha = 0 \) (Variance-gamma by Madan et al):
    \[ \psi(u) = \lambda \ln \left( 1 - iu/\beta_+ \right) \left( 1 + iu/\beta_- \right) = \lambda \left( \ln(\beta_+ - iu) - \ln \beta + \ln(\beta_- + iu) - \ln \beta_- \right). \]
  - When \( \alpha = 1 \) (exponentially dampened Cauchy, Wu 2006):
    \[ \psi(u) = -\lambda \left( (\beta_+ - iu) \ln(\beta_+ - iu) / \beta_+ + \lambda (\beta_- + iu) \ln(\beta_- + iu) / \beta_- \right) - iuC(h). \]
The Black-Scholes model

- The driver is a Brownian motion \( X_t = \sigma W_t \).
- We can write the return as
  \[
  \ln \frac{S_t}{S_0} = \mu t + \sigma W_t - \frac{1}{2} \sigma^2 t.
  \]

Note that \( \kappa(s) = \frac{1}{2} s^2 \sigma^2 \).

- The characteristic function of the return is:
  \[
  \phi(u) = \exp \left( iu \mu t - \frac{1}{2} u^2 \sigma^2 t - iu \frac{1}{2} \sigma^2 \right) = \exp \left( iu \mu t - \frac{1}{2} \sigma^2 (u^2 + iu) t \right).
  \]

Under \( \mathbb{Q} \), \( \mu = r - q \).

- The characteristic exponent of the convexity adjusted Lévy process \( (X_t - \kappa X(1)t) \) is: \( \psi_X(u) + iu \kappa_X(1) = \frac{1}{2} u^2 \sigma^2 + iu \frac{1}{2} \sigma^2 = \frac{1}{2} \sigma^2 (u^2 + iu) \).
Merton (1976)’s jump-diffusion model

- The driver of this model is a Lévy process that has both a diffusion component and a jump component.

- The Lévy triplet is \((0, \sigma, \pi(x))\), with \(\pi(x) = \lambda \frac{1}{\sqrt{2\pi}v_J} \exp \left( -\frac{(x-\mu_J)^2}{2v_J} \right) \).
  
  - The first component of the triplet (the drift) is always normalized to zero.
  
  - The characteristic exponent of the Lévy process is
    \[ \psi_X(u) = \frac{1}{2} u^2 \sigma^2 + \lambda \left( 1 - e^{iu\mu_J} - \frac{1}{2} u^2 v_J \right) \]. The cumulant exponent is
    \[ \kappa_X(s) = \frac{1}{2} s^2 \sigma^2 + \lambda \left( e^{s\mu_J} + \frac{1}{2} s^2 v_J - 1 \right) \].

- We can write the return as
  \[ \ln \frac{S_t}{S_0} = \mu t + X_t - \left( \frac{1}{2} \sigma^2 + \lambda \left( e^{\mu_J + \frac{1}{2} v_J} - 1 \right) \right) t. \]

- The characteristic function of the return is:
  \[ \phi(u) = e^{iu\mu t} e^{-\frac{1}{2} \sigma^2 (u^2 + iu)t} e^{-\left( \lambda \left( 1 - e^{iu\mu_J} - \frac{1}{2} u^2 v_J \right) + iu\lambda \left( e^{\mu_J + \frac{1}{2} v_J} - 1 \right) \right) t}. \]

- Under \( \mathbb{Q} \), \( \mu = r - q \).
Dampened power law (DPL)

- The driver of this model is a pure jump Lévy process, with its characteristic exponent 
  \[ \psi_X(u) = -\lambda \Gamma(-\alpha) \left[ (\beta_+ - iu)^\alpha - \beta_+^\alpha + (\beta_- + iu)^\alpha - \beta_-^\alpha \right] - iuC(h). \]
  The cumulant exponent is 
  \[ \kappa_X(s) = \lambda \Gamma(-\alpha) \left[ (\beta_+ - s)^\alpha - \beta_+^\alpha + (\beta_- + s)^\alpha - \beta_-^\alpha \right] + sC(h). \]

- We can write the return as, 
  \[ \ln S_t/S_0 = \mu t + X_t - \kappa_X(1)t. \]

- The characteristic function of the return is: 
  \[ \phi(u) = e^{iu\mu t} e^{-\lambda \Gamma(-\alpha) \left[ (\beta_+ - iu)^\alpha - \beta_+^\alpha + (\beta_- + iu)^\alpha - \beta_-^\alpha \right] + iu\lambda \Gamma(-\alpha) \left[ (\beta_+ - 1)^\alpha - \beta_+^\alpha + (\beta_- + 1)^\alpha - \beta_-^\alpha \right]} t. \]

- Under \( Q \), \( \mu = r - q \).

- The characteristic exponent of the convexity adjusted Lévy process \( X_t - \kappa_X(1)t \) is: 
  \[ \psi_X(u) + iu\kappa_X(1). \]

References:
Special cases of DPL

- **α-stable law**: No exponential dampening, $\beta_\pm = 0$.
  
  
  Without exponential dampening, return moments greater than $\alpha$ are no longer well defined. Characteristic function takes different form to account singularity.

  
  The characteristic exponent takes a different form as $\alpha = 0$ represents a singular point ($\Gamma(0)$ not well defined).

Other Lévy examples

- Other examples:
  - The normal inverse Gaussian (NIG) process of Barndorff-Nielsen (1998)
  - The generalized hyperbolic process (Eberlein, Keller, Prause, 1998))
  - The Meixner process (Schoutens, 2003))
  - Jump to default model (Merton, 1976)
  - ...

- Bottom line:
  - All tractable in terms of analytical characteristic exponents $\psi(u)$.
  - We can use FFT to generate the density function of the innovation (for model estimation).
  - We can also use FFT to compute option values.
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General evidence on Lévy return innovations

- Credit risk: (compound) Poisson process
  - The whole intensity-based credit modeling literature...

- Market risk: Infinite-activity jumps
  - Evidence from stock returns (CGMY (2002)): The $\alpha$ estimates for DPL on most stock return series are greater than zero.
  - Evidence from options: Models with infinite-activity return innovations price equity index options better (Carr & Wu (2003), Huang & Wu (2004))

- The role of diffusion (in the presence of infinite-variation jumps)
  - Not big, difficult to identify (CGMY (2002), Carr & Wu (2003a,b)).
  - Generate correlations with diffusive activity rates (Huang & Wu (2004)).
  - The diffusion ($\sigma^2$) is identifiable in theory even in presence of infinite-variation jumps (Aït-Sahalia (2004), Aït-Sahalia&Jacod 2005).
Implied volatility smiles & skews on a stock

AMD: 17–Jan–2006

Maturities: 32   95  186  368  732

Moneyness = \frac{\ln(K/F)}{\sigma \sqrt{\tau}}

Implied Volatility

Short–term smile

Long–term skew

Liuren Wu (Baruch)
Implied volatility skews on a stock index (SPX)

More skews than smiles

Maturities: 32  60  151  242  333  704
Average implied volatility smiles on currencies

Maturities: 1m (solid), 3m (dashed), 1y (dash-dotted)
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(I) The role of jumps at very short maturities

- Implied volatility smiles (skews) ↔ non-normality (asymmetry) for the risk-neutral return distribution.

\[ IV(d) \approx ATMV \left( 1 + \frac{\text{Skew.}}{6} d + \frac{\text{Kurt.}}{24} d^2 \right), \quad d = \frac{\ln K/F}{\sigma \sqrt{\tau}} \]

- Two mechanisms to generate return non-normality:
  - Use Lévy jumps to generate non-normality for the innovation distribution.
  - Use stochastic volatility to generates non-normality through mixing over multiple periods.

- Over very short maturities (1 period), only jumps contribute to return non-normalities.
As option maturity ↓ zero, OTM option value ↓ zero.

The speed of decay is exponential $O\left( e^{-c/T} \right)$ under pure diffusion, but linear $O(T)$ in the presence of jumps.

Term decay plot: $\ln(\text{OTM}/T) \sim \ln(T)$ at fixed $K$:

- In the presence of jumps, the Black-Scholes implied volatility for OTM options $\text{IV}(\tau, K)$ explodes as $\tau \downarrow 0$. 
The impacts of jumps at very long horizons

- Central limit theorem (CLT): Return distribution converge to normal with aggregation under certain conditions (finite return variance,...) ⇒ As option maturity increases, the smile should flatten.
- Evidence: The skew does not flatten, but steepens!
  - Return variance is infinite. ⇒ CLT does not apply.
  - Down jumps only. ⇒ Option has finite value.
- But CLT seems to hold fine statistically:
Reconcile $\mathbb{P}$ with $\mathbb{Q}$ via DPL jumps


- Model return innovations under $\mathbb{P}$ by DPL:

$$
\pi(x) = \begin{cases} 
\lambda \exp(-\beta_+ x) x^{-\alpha-1}, & x > 0, \\
\lambda \exp(-\beta_- |x|) |x|^{-\alpha-1}, & x < 0.
\end{cases}
$$

All return moments are finite with $\beta_+ > 0$. \textit{CLT applies}.

- Market price of jump risk ($\gamma$): $\frac{d\mathbb{Q}}{d\mathbb{P}} \bigg|_t = \mathcal{E}(-\gamma X)$

- The return innovation process remains DPL under $\mathbb{Q}$:

$$
\pi(x) = \begin{cases} 
\lambda \exp(-(\beta_+ + \gamma) x) x^{-\alpha-1}, & x > 0, \\
\lambda \exp(-(\beta_- - \gamma) |x|) |x|^{-\alpha-1}, & x < 0.
\end{cases}
$$

- To break CLT under $\mathbb{Q}$, set $\gamma = \beta_-$ so that $\beta_-^\mathbb{Q} = 0$.

- Reconciling $\mathbb{P}$ with $\mathbb{Q}$: \textit{Investors charge maximum allowed market price on down jumps}.
When a company defaults, its stock value jumps to zero.

It generates a steep skew in long-term stock options.

Carr and Laurence (2006) approximation of the Merton (76) jump-to-default model:

$$ IV_t(d_2, T) \approx \sigma + \frac{N(d_2)}{N'(d_2)} \sqrt{T - t} \lambda $$

The slope of the implied volatility smile at $d_2 = 0$ is $\lambda \sqrt{T - t}$.

Evidence: Stock option implied volatility skews are correlated with credit default swap (CDS) spreads written on the same company.
Three Lévy jump components in stock returns

I. Market risk (FMLS under $\mathbb{Q}$, DPL under $\mathbb{P}$)
   - The stock index skew is strongly negative at long maturities.

II. Idiosyncratic risk (DPL under both $\mathbb{P}$ and $\mathbb{Q}$)
   - The smile on single name stocks is not as negatively skewed as that on index at short maturities.

III. Default risk (Compound Poisson jumps).
   - Long-term skew moves together with CDS spreads.
   - Information and identification:
     - Identify market risk from stock index options.
     - Identify the credit risk component from the CDS market.
     - Identify the idiosyncratic risk from the single-name stock options.
Lévy jump components in currency returns

- Model currency return as the difference of the log pricing kernels between the two economies.
- Pricing kernel assigns market prices to systematic risks.
- Market risk dominates for exchange rates between two industrialized economies (e.g., dollar-euro).
  - Use a one-sided DPL for each economy (downward jump only).
- Default risk shows up in FX for low-rating economies (say, dollar-peso).
  - Peso drops by a large amount when the country (Mexico) defaults on its foreign debt.


- When pricing options on exchange rates, it is appropriate to distinguish between world risk versus country-specific risk.

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Economic implications of using jumps

- In the Black-Scholes world (one-factor diffusion):
  - The market is complete with a bond and a stock.
  - The world is risk free after delta hedging.
  - Utility-free option pricing. Options are redundant.

- In a pure-diffusion world with stochastic volatility:
  - Market is complete with one (or a few) extra option(s).
  - The world is risk free after delta and vega hedging.

- In a world with jumps of random sizes:
  - The market is inherently incomplete (with stocks alone).
  - Need all options (+ model) to complete the market.
  - Derman: “Beware of economists with Greek symbols!”
  - Options market is informative/useful:
    - Cross sections \((K, T) \leftrightarrow \mathbb{Q}\) dynamics.
    - Time series \((t) \leftrightarrow \mathbb{P}\) dynamics.
    - The difference \(\mathbb{Q}/\mathbb{P}\) \leftrightarrow market prices of economic risks.
Different types of jumps affect option pricing at both short and long maturities.

- Implied volatility smiles at very short maturities can only be accommodated by a jump component.
- Implied volatility skews at very long maturities ask for a jump process that generates infinite variance.
- Credit risk exposure may also help explain the long-term skew on single name stock options.

The choice of jump types depends on the events:

- Infinite-activity jumps $\Leftrightarrow$ frequent market order arrival.
- Finite-activity Poisson jumps $\Leftrightarrow$ rare events (credit).

The presence of jumps of random sizes creates value for the options markets...

Lévy processes are largely “static” in the sense that they cannot generate time variations in the return distribution and hence cannot accommodate stochastic volatility, stochastic skewness, etc.