Introduction to Options Markets and Behaviors

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Options Pricing
Definitions and terminologies

- An **option** gives the option holder the right/option, *but no obligation*, to buy or sell a security to the option writer/seller

  - for a pre-specified price (the **strike price**, $K$)
  - at (or up to) a given time in the future (the **expiry date**)

- An option has positive value.

  *Comparison: a forward contract has zero value at inception.*

- **Option types**

  - A **call option** gives the holder the right to buy a security. The payoff is $(S_T - K)^+$ when exercised at maturity.

  - A **put option** gives the holder the right to sell a security. The payoff is $(K - S_T)^+$ when exercised at maturity.

  - **American options** can be exercised at any time prior to expiry.

  - **European options** can only be exercised at the expiry.
More terminologies

- **Moneyness**: the strike relative to the spot/forward level
  - An option is said to be **in-the-money** if the option has positive value if exercised right now:
    - $S_t > K$ for call options and $S_t < K$ for put options.
  - Sometimes it is also defined in terms of the forward price at the same maturity (in the money forward): $F_t > K$ for call and $F_t < K$ for put.
  - The option has positive **intrinsic value** when in the money. The intrinsic value is $(S_t - K)^+$ for call, $(K - S_t)^+$ for put.
  - We can also define intrinsic value in terms of forward price.
  - An option is said to be **out-of-the-money** when it has zero intrinsic value.
    - $S_t < K$ for call options and $S_t > K$ for put options.
    - **Out-of-the-money forward**: $F_t < K$ for call and $F_t > K$ for put.
  - An option is said to be **at-the-money** spot (or forward) when the strike is equal to the spot (or forward).
More terminologies

- The value of an option is determined by
  - the current spot (or forward) price ($S_t$ or $F_t$),
  - the strike price $K$,
  - the time to maturity $\tau = T - t$,
  - the option type (Call or put, American or European), and
  - the dynamics of the underlying security (e.g., how volatile the security price is).

- Out-of-the-money options do not have intrinsic value, but they have time value (value of optionality).

- Time value is determined by time to maturity of the option and the dynamics of the underlying security.

- Generically, we can decompose the value of each option into two components:
  \[
  \text{option value} = \text{intrinsic value} + \text{time value}.
  \]
Payoffs versus P&Ls

- For European options, the terminal payoff can be written as \((S_T - K)^+\) for calls and \((K - S_T)^+\) for puts at expiry date \(T\).

- Since options have positive value, one needs to pay an upfront price (option price) to possess an option.

- The P&L from the option investment is the difference between the terminal payoff and the initial price you pay to obtain the option.

- Do not confuse the two.

- The textbook likes to talk about P&Ls, but I like to talk about payoffs — Different perspectives:
  - P&Ls: If I buy/sell an option today, how much money I can make under different scenarios? What’s my return?
  - Payoffs: If I desire a certain payoff structure in the future, what types of options/positions I need to generate it?
Consider a European call option on a stock index. The current index level (spot $S_t$) is 100. The option has a strike ($K$) of $90$ and a time to maturity ($T - t$) of 1 year. The option has a current value ($c_t$) of $14$.

- Is this option in-the-money or out-of-the-money (wrt to spot)?
- What’s intrinsic value for this option? What’s its time value?
- If you hold this option, what’s your terminal payoff?
  - What’s your terminal payoff and P&L if the index level reaches 100, 90, or 80 at the expiry date $T$?
- If you write this option and have sold it to the exchange, what does your terminal payoff look like?
  - What’s your terminal payoff and P&L if the index level reaches 100, 90, or 80 at the expiry date $T$?
Payoffs and P&Ls from long/short a call option

\[(S_t = 100, K = 90, c_t = 14)\]

Long a call pays off, \((S_T - K)^+\), bets on index price going up. Shorting a call bets on index price going down.
Another example: Put option on an exchange rate

Consider a European put option on the dollar price of pound (GBPUSD). The current spot exchange rate \( S_t \) is $1.6285 per pound. The option has a strike \( K \) of $1.61 and a time to maturity \( T - t \) of 1 year. The 1-year forward price \( F_{t,T} \) is $1.61. The dollar continuously compounding interest rate at 1-year maturity \( r_d \) is 5%. The option \( p_t \) is priced at $0.0489.

- From the above information, can you infer the continuously compounding interest rate at 1-year maturity on pound \( r_f \)?

- Is this option in-the-money or out-of-the-money wrt to spot? What’s the moneyness in terms of forward?

- In terms of forward, what’s intrinsic value for this option? What’s its time value?

- If you hold this option, what’s your terminal payoff, if the dollar price of pound reaches 1.41, 1.61, or 1.81 at the expiry date \( T \)?
Another example: Put option on an exchange rate

- Review the forward pricing formula: \( F_{t,T} = S_t e^{(r_d - r_f)(T-t)} \).
  \[ r_f = r_d - \frac{1}{T-t} \ln(F_{t,T}/S_t) = 0.05 - \ln(1.61/1.6285)/1 = 6.14\% . \]

- Recall covered interest rate parity: Annualized forward return \( (\frac{1}{T-t} \ln(F_{t,T}/S_t)) \) on exchange rates equals interest rate differential \( (r_d - r_f) \) between the two currencies.

- Long a put option pays off, \( (K - S_T)^+ \), and bets on the underlying currency (pound) depreciates.

- Shorting a put option bets on pound appreciates.

- *How does it differ from betting using forwards?*
Payoffs and P&Ls from long/short a put option

\((S_t = 1.6285, F_{t,T} = 1.61, K = 1.61, p_t = 0.0489)\)
What derivative positions generate the following payoff?
Option value behaviors: Notation

- $S$: stock price. $F_{t,T}$: time-$t$ forward price with expiry $T$.
- $K$: strike price.
- Today: either time 0 or time $t$.
- $T$: expiry date (or maturity with $t = 0$).
- $\sigma$: Volatility (annualized standard deviation) of stock return.
- $r$: continuously compounded riskfree rate with maturity $T$ (same as option).
- $D$: present value of discrete dividends paid during option’s life.
- $q$: continuously compounded dividend yield during option’s life (for foreign interest rate for currency options).
Option variation with spot price

- Dependence on spot follows from payoff function (when time to maturity is zero).

In examples given below, I use the following benchmark numbers: $S_t = 100; K = 100; \sigma = 20\%; t = 0; T = 2/12; r = 5\%; q = 3\%$. (unless otherwise specified)
Dependence on strike follows from payoff function (when time to maturity is zero).
- Normally increase, but can decline, esp. for far out of money options.
- American option value always increases with time, given the option to exercise early.
- ATM option value increases with (approximately) square root of time.
ATM option value is almost linear in volatility (square root of time).

Options market is mainly a market for volatility.
Increasing interest rate or reducing dividend yield increases the growth rate of the stock price.
American v. European options

- The above graphs are all generated for European options, based a simple option pricing model, which we’ll deal with later.

- An American option is worth at least as much as the corresponding European option: \( C \geq c \) and \( P \geq p \).
  - The difference is due to the extra option you get from an American option: You can exercise any time before the expiry date.
  - Hence, the difference is also referred to as the early exercise premium.

- Pricing American options is a bit harder — we’ll need a numerical algorithm to deal with the early exercise premium.
Obvious arbitrage opportunities on call options

- Suppose that $c_t = 2, S_t = 20, T - t = 1, K = 17, r = 5\%, q = 0$. Is there an arbitrage opportunity?

- Suppose that $c_t = 21, S_t = 20, T - t = 1, K = 1, r = 5\%, q = 0$. Is there an arbitrage opportunity?
Bounds on a call option

- **Lower bound:**
  \[ c_t \geq \max[0, S_t e^{-q(T-t)} - Ke^{-r(T-t)}] = \max[0, e^{-r(T-t)}(F_{t,T} - K)] . \]
  - If the call price is negative, you are paid to own an option.
  - If the call price is lower than the value of the forward at the same strike, \( e^{-r(T-t)}(F_{t,T} - K) \), buy the call and short the forward:
    - Today’s value from the transaction is: \(-c_t + e^{-r(T-t)}(F_{t,T} - K) > 0\).
    - At maturity, the payoff is \((S_T - K)^+ - (S_T - K) = (K - S_T)^+ > 0\).
  - Point to remember: *An option is worth more than a forward.*

- **Upper bound:**
  \[ c_t \leq S_t e^{-q(T-t)} = e^{-r(T-t)}F_{t,T} . \]
  - If violated, write the call and long a forward at zero strike.
    - Today’s value from the transaction is: \(c_t - e^{-r(T-t)}F_{t,T} > 0\).
    - At maturity, the payoff is \(-(S_T - K)^+ + S_T = \min(S_T, K) > 0\).
  - A forward at zero strike is the same as a call at zero strike.
  - A call with positive strike is worth less than a call with zero strike.
  - More generally, \( c(K_1) \leq c(K_2) \) for all \( K_1 \geq K_2 \). — Can you devise an arbitrage trading when this *monotonicity condition* is violated?

- In the presence of discrete dividends, the bounds are:
  \[ [S_t - D - Ke^{-r(T-t)}, S_t - D] . \]
Suppose that $c_t = 3.2, S_t = 20, T - t = 1, K = 17, r = 5\%, q = 3\%$. Is there an arbitrage opportunity?

$$S_t e^{-q(T-t)} - Ke^{-r(T-t)} = 3.238.$$ 

Suppose that $c_t = 19.5, S_t = 20, T - t = 1, K = 1, r = 5\%, q = 3\%$. Is there an arbitrage opportunity?

$$S_t e^{-q(T-t)} = 19.41.$$ 

Suppose that $c_t = 19.5, S_t = 20, T - t = 1, K = 1, r = 5\%, D = 1$. Is there an arbitrage opportunity?

$$S_t - D = 19.$$
Bounds on a put option

- **Lower bound:**
  \[ p_t \geq \max[0, Ke^{-r(T-t)} - S_t e^{-q(T-t)}] = \max[0, e^{-r(T-t)}(K - F_{t,T})]. \]
  - If put price is negative, you are paid to own an option.
  - If put price is lower than the value of the corresponding short forward, buy the put and long the forward:
    - Today's value from the transaction is: \(-p_t - e^{-r(T-t)}(F_{t,T} - K) > 0.\)
    - At maturity, the payoff is \((K - S_T)^+ + (S_T - K) = (S_T - K)^+ > 0.\)
  - *An option to sell is worth more than a forward to sell.*
  - With discrete dividends, the lower bound is: \(Ke^{-r(T-t)} - S_t + D.\)

- **Upper bound:** \(p_t \leq Ke^{-r(T-t)}.\)
  - If violated, write the put and save \(Ke^{-r(T-t)}\) in the bank. Spend the rest at water park.
    - Today's value: water slides.
    - At maturity, the return from the bank is \(K\). The worst possible obligation from writing the put is \(K\) when the company goes bankruptcy \((S_T = 0)\). So you make even in the worst case, make money otherwise. \(K - (K - S_T)^+ = \min(S_T, K) > 0.\)
  - **Monotonicity condition:** \(p(K_1) \geq p(K_2)\) for all \(K_1 \geq K_2.\)
Suppose that $p_t = 2, S_t = 20, T - t = 1, K = 23, r = 5\%, q = 3\%$. Is there an arbitrage opportunity?

$$Ke^{-r(T-t)} - S_t e^{-q(T-t)} = 2.47.$$

Suppose that $p_t = 2, S_t = 20, T - t = 1, K = 23, r = 5\%, D = 1$. Is there an arbitrage opportunity?

$$Ke^{-r(T-t)} - S_t + D = 2.88.$$

Suppose the put price at $20$ strike is $2$ and at $21$ strike is $1.9$ at the same maturity. Is there an arbitrage opportunity?
Recall the put-call parity condition:
The difference between a call and a put equals the forward.

\[ c_t - p_t = e^{-r(T-t)}(F_t, T - K_t) \]
\[ = S_t e^{-q(T-t)} - K_t e^{-r(T-t)} \]
\[ = S_t - D - K_t e^{-r(T-t)} \]

Suppose that \( c_t = 8, S_t = 100, T - t = 1, K = 100, r = 0, q = 0 \).
Is there an arbitrage opportunity if

- \( p_t = 7 \).
- \( p_t = 8 \).
- \( p_t = 9 \).
Early exercise of American options

- Usually there is some chance that an American option will be exercised early.
  - Exercise when the “exercise value” \((S_t - K)^+\) for call) is higher than the option value \((C_t)\).
- As a result, the American option is more expensive (valuable) than the European option.
- An exception is an American call on a non-dividend paying stock, which should never be exercised early.
  - \(C_t \geq c_t \geq S_t - Ke^{-r(T-t)} \geq S_t - K = \text{exercise value}\)
  - The call option value for a non-dividend paying stock is always no lower than the exercise value.
Cheat sheet: Summary of important relations/bounds

- **Forward pricing:**
  \[ F_t = e^{(r-q)(T-t)} S_t = e^{r(T-t)}(S_t - D_t), \]

- **European options bounds:**
  \[ c_t \in [e^{-r(T-t)}(F_t, T - K)^+, \ e^{-r(T-t)}F_t, T ] \]
  \[ p_t \in [e^{-r(T-t)}(K - F_t, T)^+, \ e^{-r(T-t)}K ] \]

- **Put-call parity for European options:**
  \[ c_t - p_t = e^{-r(T-t)}(F_t, T - K_t) = e^{-q(T-t)}S_t - e^{-r(T-t)}K. \]

- **Put-call inequality for American options:**
  \[ S_t - D - K \leq C_t - P_t \leq S_t - Ke^{-r(T-t)} \]
  \[ S_t e^{-q(T-t)} - K \leq C_t - P_t \leq S_t - Ke^{-r(T-t)} \]

- **Monotonicity for both European and American options:**
  \[ p(K_1) \geq p(K_2), \ c(K_1) \leq c(K_2), \ \text{for all} \ K_1 > K_2. \]

- **Most important:** Know how to trade against violations of these conditions.
Basic terminologies: call, put, American, European, in-the-money, out-of-the-money, intrinsic value, time value...

Basic mechanisms of options trading: market making, margins, exchanges, stock splits, ...

Inside-out knowledge on payoff structures of different positions (long/short) in different derivatives (call/put, forward, spot).

Option behaviors: How does option value vary with strike, spot, volatility, and time.

Option no-arbitrage bounds.