Long Term Risk: A Martingale Approach

Likuan Qin* and Vadim Linetsky†

Department of Industrial Engineering and Management Sciences
McCormick School of Engineering and Applied Sciences
Northwestern University

Abstract

This paper extends the long-term factorization of the pricing kernel due to Alvarez and Jermann (2005) in discrete time ergodic environments and Hansen and Scheinkman (2009) in continuous ergodic Markovian environments to general semimartingale environments, without assuming the Markov property. An explicit and easy to verify sufficient condition is given that guarantees convergence in Emery’s semimartingale topology of the trading strategies that invest in $T$-maturity zero-coupon bonds to the long bond and convergence in total variation of $T$-maturity forward measures to the long forward measure. As applications, we explicitly construct long-term factorizations in generally non-Markovian Heath-Jarrow-Morton (1992) models evolving the forward curve in a suitable Hilbert space and in the non-Markovian model of social discount of rates of Brody and Hughston (2013). As a further application, we extend Hansen and Jagannathan (1991), Alvarez and Jermann (2005) and Bakshi and Chabi-Yo (2012) bounds to general semimartingale environments. When Markovian and ergodicity assumptions are added to our framework, we recover the long-term factorization of Hansen and Scheinkman (2009) and explicitly identify their distorted probability measure with the long forward measure. Finally, we give an economic interpretation of the recovery theorem of Ross (2013) in non-Markovian economies as a structural restriction on the pricing kernel leading to the growth optimality of the long bond and identification of the physical measure with the long forward measure. This latter result extends the interpretation of Ross’ recovery by Borovička et al. (2014) from Markovian to general semimartingale environments.

1 Introduction

The stochastic discount factor (SDF) assigns today’s prices to risky future payoffs at alternative investment horizons. It accomplishes this by simultaneously discounting the future and adjusting for risk (the reader is referred to Hansen (2013) and Hansen and Renault (2009) for surveys of SDFs). One useful decomposition of the SDF that explicitly features its discounting and risk adjustment functions is the factorization of the SDF into the risk-free discount factor discounting

*likuanqin2012@u.northwestern.edu
†linetsky@iems.northwestern.edu
at the (stochastic) riskless rate and a positive unit-mean martingale encoding the risk premium. The martingale can be conveniently used to change over to the risk-neutral probabilities (Cox and Ross (1976a), Cox and Ross (1976b), Harrison and Kreps (1979), Harrison and Pliska (1981), Delbaen and Schachermayer (2006)). Under the risk-neutral probability measure risky future payoffs are discounted at the riskless rate, and no explicit risk adjustment is required, as it is already encoded in the risk-neutralized dynamics. In other words, the riskless asset (often called riskless savings account) serves as the numeraire asset under the risk-neutral measure, making prices of other assets into martingales when re-denominated in units of the savings account.

An alternative decomposition of the SDF arises if one discounts at the rate of return on the $T$-maturity zero-coupon bond. The corresponding factorization of the SDF features a martingale that accomplishes a change to the so-called $T$-forward measure (Jarrow (1987), Geman (1989), Jamshidian (1989), Geman et al. (1995)). The $T$-maturity zero-coupon bond serves as the numeraire asset under the $T$-forward measure.

More recently Alvarez and Jermann (2005), Hansen et al. (2008), and Hansen and Scheinkman (2009) introduce and study an alternative factorization of the SDF, where one discounts at the rate of return on the zero-coupon bond of asymptotically long maturity (often called the long bond). The resulting long-term factorization of the SDF features a martingale that accomplishes a change of probabilities to a new probability measure under which the long bond serves as the numeraire asset. Combining the risk-neutral and the long-term decomposition reveals how the SDF discounts future payoffs at the riskless rate of return, a spread that features the premium for holding the long bond, and a martingale component encoding an additional time-varying risk premium. Alvarez and Jermann (2005) introduce the long-term factorization in discrete-time, ergodic environments. Hansen and Scheinkman (2009) provide far reaching extensions to continuous ergodic Markovian environments and connect the long-term factorization with positive eigenfunctions of the pricing operator, among other results. Hansen (2012), Hansen and Scheinkman (2012a), Hansen and Scheinkman (2012b), Hansen and Scheinkman (2014), and Borovička et al. (2014) develop a wide range of economic applications of the long-term factorization. Bakshi and Chabi-Yo (2012) empirically estimate bounds on the factors in the long-term factorization using US data, complementing original empirical results of Alvarez and Jermann (2005).

Contributions of the present paper are as follows.

- We give a formulation of the long-term factorization in general semimartingale environments, dispensing with the Markovian assumption underpinning the work of Hansen and Scheinkman (2009). We start by assuming that there is a strictly positive, integrable semimartingale pricing kernel $S$ with the process $S_-$ of its left limits also strictly positive. The key result of this paper is Theorem 4.1 that specifies an explicit and easy to verify sufficient condition that ensures that the wealth processes of trading strategies investing in $T$-maturity zero-coupon bonds and rolling over as bonds mature at time intervals of duration $T$ converge to the long bond in Emery’s semimartingale topology. Furthermore, we prove that, under our condition, the resulting probability measure is the limit in total variation of the $T$-maturity forward measures. We thus call it the long forward measure.

- As an application, we verify our sufficient condition in a class of Heath, Jarrow and Morton (1992) models with forward rate curves taking values in an appropriate Hilbert space and driven by a (generally infinite-dimensional) Wiener process and obtain explicitly the long-
term factorization of the pricing kernel in non-Markovian HJM models. This application illustrates the scope and generality of our results relative to the original results of Hansen and Scheinkman (2009) focused on ergodic Markovian environments.

- As another example, we verify our sufficient condition and obtain explicitly the long-term factorization in the non-Markovian model of Brody and Hughston (2013) of social discount rates.

- As a further application, we extend Hansen and Jagannathan (1991), Alvarez and Jermann (2005) and Bakshi and Chabi-Yo (2012) bounds to general semimartingale environments.

- Furthermore, when we do impose Markovian and ergodicity assumptions in our framework, we recover the long-term factorization of Hansen and Scheinkman (2009) and explicitly identify their distorted probability measure associated with the principal eigenfunction of the pricing operator with the long forward measure. Our sufficient condition for existence of the long bond and, thus, for existence of the long-term factorization, when restricted to Markovian environments, turns out to be distinct from the conditions in Hansen and Scheinkman (2009). Interestingly, it does not require ergodicity, as in Hansen and Scheinkman (2009). Indeed, we provide an explicit example of an affine model with absorption at zero, where the underlying Markovian driver is transient, but the long-term factorization nevertheless exists.

- Viewed through the prism of our results in semimartingale environments, the issue of uniqueness studied in Hansen and Scheinkman (2009) does not arise, as the long-term factorization is unique by our definition as the long term limit of the $T$-forward factorizations.

- We give some further explicit sufficient conditions for existence of the long-term factorization in specific Markovian environments that are easier to verify in concrete model specifications than either our general sufficient condition in semimartingale environments or sufficient conditions in Markovian environments due to Hansen and Scheinkman (2009).

- As an immediate application, we give an economic interpretation and an extension of the Recovery Theorem of Ross (2013) to general semimartingale environments (from Ross’ original formulation for discrete-time, finite-state irreducible Markov chains and Qin and Linetsky (2014) formulation for continuous-time Markov processes). Namely, we re-interpret Ross’ main assumption as the assumption that the long bond is growth optimal. This then immediately results in the identification of the physical measure with the long forward measure. This extends the economic interpretations of Ross’ recovery offered by Martin and Ross (2013) in discrete-time Markov chain environments and by Borovička et al. (2014) in continuous Markovian environments to general semimartingale environments.

The main message of this paper is that the long-term factorization is a general phenomenon featured in any arbitrage-free, frictionless economy with a positive semimartingale pricing kernel that satisfies the sufficient condition of Theorem 4.1, and is not limited to Markovian environments.

The rest of this paper is organized as follows. In Section 2 we define our semimartingale environment, formulate our assumptions on the pricing kernel, and define zero-coupon bonds and forward measures. In Section 3 we briefly recall the risk-neutral factorization. In Section
4 we formulate our central result, Theorem 4.1, that gives a sufficient condition for existence of the long bond and the long forward measure. In Section 5 we present extensions of Hansen and Jagannathan (1991), Alvarez and Jermann (2005) and Bakshi and Chabi-Yo (2012) bounds to our general semimartingale environment. In Section 6 we apply Theorem 4.1 to explicitly obtain the long-term factorization in non-Markovian Heath-Jarrow-Morton models. In Section 7 we apply Theorem 4.1 to explicitly obtain the long-term factorization in non-Markovian model of Brody and Hughston (2013) of social discount rates. Section 8 studies the Markovian environment. Section 9 gives an economic interpretation of Ross’ recovery without the Markov property. Proofs and supporting technical results are collected in appendices.

2 Semimartingale Pricing Kernels and Forward Measures $Q^T$

We work on a complete filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, \mathbb{P})$ with the filtration $(\mathcal{F}_t)_{t \geq 0}$ satisfying the usual conditions of right continuity and completeness. We assume that $\mathcal{F}_0$ is trivial modulo $\mathbb{P}$. All random variables are identified up to almost sure equivalence. We choose all semimartingales to be right continuous with left limits (RCLL) without further mention (the reader is referred to Jacod and Shiryaev and Protter for stochastic calculus of semimartingales; a summary of some key results can also be found in Jacod and Protter (2003)). For a RCLL process $X$, $X^-$ denotes the process of its left limits, $(X^-)_t = \lim_{s \to t, s < t} X_s$ for $t > 0$ and $(X^-)_0 = X_0$ by convention (cf. Jacod and Shiryaev (2003)). We assume absence of arbitrage and trading frictions so that there exists a pricing kernel (state-price density) process $S_t = (S_t)_{t \geq 0}$ satisfying the following assumptions.

Assumption 2.1. (Semimartingale Pricing Kernel) The pricing kernel process $S$ is a strictly positive semimartingale with $S_0 = 1$, $S^-$ is strictly positive, and $E^\mathbb{P}[S_T/S_t] < \infty$ for all $T > t \geq 0$.

For each pair of times $t$ and $T$, $0 \leq t \leq T < \infty$, the pricing kernel defines a family of pricing operators $(P_{t,T})_{0 \leq t \leq T}$ mapping time-$T$ payoffs $Y$ ($\mathcal{F}_T$-measurable random variables) into their time-$t$ prices $P_{t,T}(Y)$ ($\mathcal{F}_t$-measurable random variables):

$$P_{t,T}(Y) = E^\mathbb{P}\left[\frac{S_T Y}{S_t} \bigg| \mathcal{F}_t\right], \quad t \in [0, T].$$

where $S_T/S_t$ is the stochastic discount factor from $T$ to $t$. $P_{t,T}$ are positive, linear, and satisfy time consistency

$$P_{s,t}(P_{t,T}(Y)) = P_{s,T}(Y)$$

for $Y \in \mathcal{F}_T$.

In particular, a $T$-maturity zero-coupon bond has a unit cash flow at time $T$ and a price process

$$P^T_t = P_{t,T}(1) = E^\mathbb{P}\left[\frac{S_T}{S_t} \bigg| \mathcal{F}_t\right], \quad s \leq t \leq T.$$ 

Under our assumptions, for each maturity $T > 0$ the zero-coupon bond price process $(P^T_t)_{t \in [0,T]}$ is a positive semimartingale such that $P^T_T = 1$ and the process $(M^T_t := S_t P^T_t / P^T_0)_{t \in [0,T]}$ is a positive martingale (our integrability assumption on the stochastic discount factor implies that
$\mathbb{E}^{P}[P_{t}^{T}] < \infty$ for all future times $t$ and bond maturities $T > t$). For each $T$, we can thus write a factorization for the pricing kernel on the time interval $[0, T]$: 

$$S_t = \frac{P_{0}^{T}}{P_{t}^{T}} M_{t}^{T}. $$

We can then use the martingale $M_{t}^{T}$ to define a new probability measure

$$Q_{t}^{T} = M_{t}^{T} \mathbb{P} |_{\mathcal{F}_{t}}, \ t \in [0, T].$$

This is the $T$-forward measure originally discovered by Jarrow (1987) and later independently by Geman (1989) and Jamshidian (1989) (see also Geman et al. (1995)). Under $Q_{T}^{T}$ the $T$-maturity zero-coupon bond serves as the numeraire, and the pricing operator reads:

$$P_{s,t}^{T}(Y) = \mathbb{E}^{Q_{T}^{T}} \left[ \frac{P_{t}^{T}}{P_{s}^{T}} Y \bigg| \mathcal{F}_{s} \right]$$

for $Y \in \mathcal{F}_{t}$ and $s \leq t \leq T$.

The forward measure is defined on $\mathcal{F}_{t}$ for $t \leq T$. We now extend it to $\mathcal{F}_{t}$ for all $t \geq 0$ as follows. Fix $T$ and consider a self-financing roll-over strategy that starts at time zero by investing one unit of account in $\frac{1}{P_{0}^{T}}$ units of the $T$-maturity zero-coupon bond. At time $T$ the bond matures, and the value of the strategy is $\frac{1}{P_{0}^{T}}$ units of account. We roll the proceeds over by re-investing into $\frac{1}{P_{0}^{T} P_{0}^{2T}}$ units of the zero-coupon bond with maturity $2T$. We continue with the roll-over strategy, at each time $kT$ re-investing into the bond $P_{kT}^{T+1}$. We denote the wealth process of this self-financing strategy $B_{kT}^{T}$:

$$B_{kT}^{T} = \frac{P_{t}^{(k+1)T}}{\prod_{i=0}^{k} P_{iT}^{(i+1)T}}, \ t \in [kT, (k+1)T) \quad k = 0, 1, \ldots.$$ 

It is clear by construction that the process $S_t B_{k}^{T}$ extends the martingale $M_{t}^{T}$ to all $t \geq 0$, and, thus, Eq.(2.1) defines the $T$-forward measure on $\mathcal{F}_{t}$ for all $t \geq 0$, where $T$ now has the meaning of duration of the compounding interval. Under the forward measure $Q_{T}^{T}$ extended to all $\mathcal{F}_{t}$, the roll-over strategy $(B_{t}^{T})_{t \geq 0}$ with compounding interval $T$ serves as the new numeraire. We continue to call the measure extended to all $\mathcal{F}_{t}$ for $t \geq 0$ the $T$-forward measure and use the same notation, as it reduces to the standard definition of the forward measure on $\mathcal{F}_{t}$ for $t \in [0, T]$.

Since the roll-over strategy $(B_{t}^{T})_{t \geq 0}$ and the positive martingale $M_{t}^{T} = S_{t} B_{t}^{T}$ are now defined for all $t \geq 0$, we can write the $T$-maturity factorization of the pricing kernel:

$$S_t = \frac{1}{B_t^{T}} M_{t}^{T}, \quad t \geq 0. $$

We will now investigate two limits, the short-term limit $T \to 0$ and the long-term limit $T \to \infty$.

3 Short-term Limit $T \to 0$, Implied Savings Account, and Risk Neutral Measure

In this Section we also assume that $S$ is a special semimartingale, i.e. the process of finite variation in the semimartingale decomposition of $S$ into the sum of a local martingale and a
process of finite variation can be taken to be predictable (this assumption is only made in this Section and is not made in the rest of the paper). Since both \( S \) and \( S^- \) are strictly positive, \( S \) admits a unique factorization (cf. Jacod (1979), Proposition 6.19 or Döblerlein and Schweizer (2001), Proposition 2):
\[
S_t = \frac{1}{A_t} M_t, \tag{3.1}
\]
where \( A \) is a strictly positive predictable process of finite variation with \( A_0 = 1 \) and \( M \) a strictly positive local martingale with \( M_0 = 1 \).

**Assumption 3.1. (Existence of Implied Savings Account)** The local martingale \( M \) in the factorization Eq. (3.1) of the pricing kernel is a true martingale.

Under this assumption, we can define a new probability measure \( \mathbb{Q}|_{\mathcal{F}_t} = M_t \mathbb{P}|_{\mathcal{F}_t} \) equivalent to \( \mathbb{P} \) on each \( \mathcal{F}_t \). Under \( \mathbb{Q} \) the pricing operators read:
\[
\mathbb{P}_{t,T}(Y) = \mathbb{E}^Q \left[ \frac{A_t}{A_T} Y \right] \bigg| \mathcal{F}_t.
\]
The process \( A \) is called *implied savings account* implied by the term structure of zero-coupon bonds \( (P^T_t)_{0 \leq t \leq T} \) (see Döblerlein and Schweizer (2001) and Döblerlein et al. (2000) for a definitive treatment in the semimartingale framework, and Rutkowski (1996) and Musiela and Rutkowski (1997) for earlier references). The name implied savings account is justified since \( A \) is a predictable process of finite variation. If either it is directly assumed that there is a tradeable asset with this price process, or it can be replicated by a self-financing trading strategy in other tradeable assets, then this asset or self-financing strategy is *locally* riskless due to the lack of a martingale component. Under either assumption, \( \mathbb{Q} \) plays the role of the risk-neutral measure with the implied savings account \( A \) serving as the numeraire asset.

Furthermore, Döblerlein and Schweizer (2001) explicitly show that under some additional technical conditions the implied savings account \( A \) can, in fact, be identified with the limit \( B^0 \) of self-financing roll-over strategies \( B^T \) as the roll-over interval \( T \) goes to zero, which they call the *classical savings account* (their analysis is in turn based on the work of Björk et al. (1997) who also investigate the roll-over strategy). We can then identify the risk-neutral measure \( \mathbb{Q} \) using the implied savings account as the numeraire with the measure \( \mathbb{Q}^0 \) using the limiting roll-over strategy as \( T \to 0 \) (classical savings account) as the numeraire.

If we assume that the pricing kernel \( S \) is a supermartingale, then, by multiplicative Doob-Meyer decomposition (cf. part 2 of Proposition 2 in Döblerlein and Schweizer (2001)), the implied savings account \( A \) is non-decreasing. It is then globally riskless (in the sense of no decline in value). If we assume that the implied savings account \( A \) is absolutely continuous, we can write
\[
A_t = e^{\int_0^t r_s \, ds},
\]
where \( r \) is the short-term interest rate process (short rate). If we assume that both \( S \) is a supermartingale and the implied savings account \( A \) is absolutely continuous, then the short rate is non-negative. In this paper, unless explicitly stated otherwise in examples, we generally neither assume that \( A \) is non-decreasing nor that it is absolutely continuous (i.e. that the short rate exists). In fact, in Section 4 we do not generally assume that the implied savings account exists, i.e. we do not generally make Assumption 3.1 (But Assumption 3.1 together with the absolute continuity of \( A \) is automatically satisfied under assumptions of Section 6 in HJM models).
4 The Long-term Limit $T \to \infty$, Long Bond, and Long Forward Measure

The $T$-forward factorization (2.2) decomposes the pricing kernel into discounting at the rate of return on the $T$-maturity bond and a further risk adjustment. The short-term factorization (3.1) emerging in the short-term limit $T \to 0$ as discussed in the previous section decomposes the pricing kernel into discounting at the rate of return on the riskless asset and adjusting for risk. In contrast, in this section our main focus is the long-term limit $T \to \infty$.

**Definition 4.1. (Long Bond)** If the wealth processes $(B^T_t)_{t \geq 0}$ of the roll-over strategies in $T$-maturity bonds converge to a strictly positive semimartingale $(B^\infty_t)_{t \geq 0}$ uniformly on compacts in probability as $T \to \infty$, i.e. for all $t > 0$ and $K > 0$

$$\lim_{T \to \infty} \mathbb{P}(\sup_{s \leq t} |B^T_s - B^\infty_s| > K) = 0,$$

we call the limit the long bond.

**Definition 4.2. (Long Forward Measure)** If there exists a measure $\mathbb{Q}^\infty$ locally equivalent to $\mathbb{P}$ such that the $T$-forward measures converge strongly to $\mathbb{Q}^\infty$ on each $\mathcal{F}_t$, i.e.

$$\lim_{T \to \infty} \mathbb{Q}^T(A) = \mathbb{Q}^\infty(A)$$

for each $A \in \mathcal{F}_t$ and each $t \geq 0$, we call the limit the long forward measure and denote it $\mathbb{L}$.

The following theorem is the central result of this paper. It gives an explicit sufficient condition easy to verify in applications that ensures stronger modes of convergence (Emery’s semimartingale convergence to the long bond and convergence in total variation to the long forward measure).

**Theorem 4.1. (Long-Term Factorization of the Pricing Kernel)** Suppose that for each $t > 0$ the sequence of strictly positive random variables $M^T_t$ converge in $L^1$ to a strictly positive limit:

$$\lim_{T \to \infty} \mathbb{E}^\mathbb{P}[|M^T_t - M^\infty_t|] = 0 \quad (4.1)$$

with $M^\infty_t > 0$ a.s. Then the following results hold.

(i) $(M^\infty_t)_{t \geq 0}$ is a positive $\mathbb{P}$-martingale and $(M^T_t)_{t \geq 0}$ converge to $(M^\infty_t)_{t \geq 0}$ in the semimartingale topology.

(ii) Long bond $(B^\infty_t)_{t \geq 0}$ exists and $(B^T_t)_{t \geq 0}$ converge to $(B^\infty_t)_{t \geq 0}$ in the semimartingale topology.

(iii) The pricing kernel possesses a factorization

$$S_t = \frac{1}{B^\infty_t} M^\infty_t. \quad (4.2)$$

(iv) $T$-forward measures $\mathbb{Q}^T$ converge to the long forward measure $\mathbb{L}$ in total variation on each $\mathcal{F}_t$, and

$$\mathbb{L}|_{\mathcal{F}_t} = M^\infty_t \mathbb{P}|_{\mathcal{F}_t}$$

for each $t \geq 0$. 

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The proof of Theorem 4.1 is given in Appendix A. The value of the long bond at time \( t \), \( B^\infty_t \), has the interpretation of the gross return earned starting from time zero up to time \( t \) on holding the zero-coupon bond of asymptotically long maturity. The long-term factorization of the pricing kernel (4.2) into discounting at the rate of return on the long bond and a martingale component encoding a further risk adjustment extends the long-term factorization of Alvarez and Jermann (2005) and Hansen and Scheinkman (2009) to general semimartingale environments.

**Remark 4.1.** By Theorem II.5 in Memin (1980), Emery’s semimartingale convergence is invariant under locally equivalent measure transformations. Furthermore, Eq. (4.1) can be written under any locally equivalent probability measure. Namely, let \( V_t \) be a strictly positive semimartingale with \( V_0 = 1 \) and such that \( S_t V_t \) is a martingale, and define \( Q^V \) for all \( t \geq 0 \).

\[
\lim_{T \to \infty} \mathbb{E}^{Q^V} \left[ \frac{B^T_t}{V_t} - \frac{B^\infty_t}{V_t} \right] = 0. \tag{4.3}
\]

In applications it helps us choose a convenient measure \( Q^V \) to verify this condition in a concrete model.

We now consider a special class of pricing kernels.

**Definition 4.3.** We say that an economy is long-term risk-neutral if the condition in Theorem 4.1 holds and the pricing kernel has the form

\[ S_t = \frac{1}{B^\infty_t} \]

for all \( t \geq 0 \).

By definition, in a long-term risk-neutral economy \( M^\infty_t = 1 \), \( \mathbb{P} = \mathbb{L} \) and we immediately have

**Proposition 4.1.** In a long-term risk-neutral economy the long bond is growth optimal (i.e. it has the highest expected log return).

**Proof.** Consider the value process \( X_t \) of an asset or a self-financing trading strategy with \( X_0 = 1 \) and such that \( S_t X_t = X_t/B^\infty_t \) is a \( \mathbb{P} \) martingale. Thus, \( \mathbb{E}[X_t/B^\infty_t] = 1 \). By Jensen’s inequality, \( \mathbb{E}[\log(X_t/B^\infty_t)] \leq \log \mathbb{E}[X_t/B^\infty_t] = 0 \), which implies \( \mathbb{E}[\log(X_t)] \leq \mathbb{E}[\log(B^\infty_t)] \), i.e., the long bond has the highest expected log return. \( \square \)

## 5 Pricing Kernel Bounds

We now formulate Hansen and Jagannathan (1991), Alvarez and Jermann (2005) and Bakshi and Chabi-Yo (2012) bounds in our general semimartingale environment. The celebrated Hansen and Jagannathan (1991) theorem states that the ratio of the standard deviation of a stochastic discount factor to its mean exceeds the Sharpe ratio attained by any portfolio. This result holds in our general semimartingale framework. For any \( T > t \geq 0 \) and an \( \mathcal{F}_t \)-measurable random variable \( X \), let \( \sigma_t(X) := \sqrt{\mathbb{E}_t[X^2] - (\mathbb{E}_t[X])^2} \) and \( \sigma(X) := \sqrt{\mathbb{E}[X^2] - (\mathbb{E}[X])^2} \) denote its conditional and unconditional standard deviation. To formulate Alvarez and Jermann (2005) bounds, let \( L_t(X) := \log \mathbb{E}_t[X] - \mathbb{E}_t[\log X] \) and \( L(X) := \log \mathbb{E}[X] - \mathbb{E}[\log X] \) denote the conditional and unconditional Thiel’s second entropy measure, respectively, where \( \mathbb{E}_t[\cdot] = \mathbb{E}^\mathbb{P}[\cdot|\mathcal{F}_t] \).
Theorem 5.1. (Hansen-Jagannathan Bounds) Let $V_t$ be a semimartingale such that $S_t V_t$ is a martingale. Then the following inequalities hold.

(i) $\sigma \left( \frac{S_T}{S_t} \right) / \mathbb{E} \left[ \frac{S_T}{S_t} \right] \geq \left| \mathbb{E} \left[ \frac{V_T}{V_t} \right] - \frac{1}{\mathbb{E} \left[ P_T^t \right]} \right| / \sigma \left( \frac{V_T}{V_t} \right)$.

(ii) $\sigma_t \left( \frac{S_T}{S_t} \right) / \mathbb{E}_t \left[ \frac{S_T}{S_t} \right] \geq \left| \mathbb{E}_t \left[ \frac{V_T}{V_t} \right] - \frac{1}{P_T^t} \right| / \sigma_t \left( \frac{V_T}{V_t} \right)$.

The proof is based only on the martingality of $S_t V_t$ and Holder’s inequality and is given in Appendix B. If $\sigma(S_T/S_t) = \infty$ and/or $\sigma(V_T/V_t) = \infty$, the bound holds trivially. When $t = 0$, the right-hand side of (i) is the Sharpe ratio of a self-financing portfolio with the wealth process $V$ (ratio of the excess return on the portfolio over the risk-free return on investing in the $T$-maturity zero-coupon bond relative to the standard deviation of the portfolio). When $t > 0$, it can be interpreted as the forward Sharpe ratio over the future time interval $[t, T]$ as seen from time zero. The right hand side of (ii) can be interpreted as the conditional Sharpe ratio at time $t$.

The next result extends Bakshi and Chabi-Yo (2012) bounds on the factors in the long-term factorization. This result can be viewed as an elaboration on Hansen-Jagannathan type bounds for the two factors in the long-term factorization, yielding Hansen-Jagannathan type bounds for all self-financing portfolios with wealth processes $V_t$ such that $S_t V_t$ are martingales.

Theorem 5.2. (Bakshi-Chabi-Yo Bounds) Suppose the pricing kernel admits a long-term factorization Eq. (4.2) with the positive martingale $M^\infty$. Let $V_t$ be a semimartingale such that $S_t V_t$ is a martingale. Then the following inequalities hold.

(i) $\sigma \left( \frac{M_T^\infty}{M_t^\infty} \right) \geq \mathbb{E} \left[ \frac{V_T/V_t}{B_T^\infty/B_t^\infty} \right] - 1 \left/ \sigma \left( \frac{V_T/V_t}{B_T^\infty/B_t^\infty} \right) \right.$.

(ii) $\sigma \left( \frac{B_T^\infty}{B_t^\infty} \right) \geq \left( \mathbb{E} \left[ \frac{B_T^\infty}{B_t^\infty} \right] \right) \mathbb{E} \left[ \frac{B_T^\infty}{B_t^\infty} \right] - 1 / \sigma \left( \frac{B_T^\infty}{B_t^\infty} \right)$.

(iii) $\sigma \left( \frac{M_T^\infty B_T^\infty}{M_t^\infty B_t^\infty} \right) \geq \mathbb{E} \left[ \frac{V_T/V_t}{(B_T^\infty/B_t^\infty)^2} \right] - 1 / \sigma \left( \frac{V_T/V_t}{(B_T^\infty/B_t^\infty)^2} \right)$.

(iv) $\sigma_t \left( \frac{M_T^\infty}{M_t^\infty} \right) \geq \mathbb{E}_t \left[ \frac{V_T/V_t}{B_T^\infty/B_t^\infty} \right] - 1 / \sigma_t \left( \frac{V_T/V_t}{B_T^\infty/B_t^\infty} \right)$.

(v) $\sigma_t \left( \frac{B_T^\infty}{B_t^\infty} \right) \geq \left( \mathbb{E}_t \left[ \frac{1}{B_t^\infty} \right] \right) \mathbb{E}_t \left[ \frac{B_T^\infty}{B_t^\infty} \right] - 1 / \sigma_t \left( \frac{B_T^\infty}{B_t^\infty} \right)$.

(vi) $\sigma_t \left( \frac{M_T^\infty B_T^\infty}{M_t^\infty B_t^\infty} \right) \geq \mathbb{E}_t \left[ \frac{V_T/V_t}{(B_T^\infty/B_t^\infty)^2} \right] - 1 / \sigma_t \left( \frac{V_T/V_t}{(B_T^\infty/B_t^\infty)^2} \right)$.

The proof is essentially the same as in for the Hansen-Jagannathan bounds, is based only on the martingale property and Holder’s inequality, and is given in Appendix B. If either variance is infinite, the bounds hold trivially. Bakshi and Chabi-Yo (2012) estimate their bounds for the volatility of the permanent (martingale) and transient components in the long-term factorization from empirical data on returns on multiple asset classes. Just as the original Hansen-Jagannathan bounds, the economic power of these bounds is their applicability to all self-financing portfolios in an arbitrage-free model such that $S_t V_t$ is a martingale.

Our next result extends Alvarez and Jermann (2005) bounds on the martingale component in the long-term factorization.
Theorem 5.3. (Alvarez-Jermann Bounds) Suppose the pricing kernel admits a long-term factorization Eq. (4.2) with the positive martingale $M_t^\infty$. Let $V_t$ be a semimartingale such that $S_tV_t$ is a martingale. Then the following inequalities hold.

(i) $L\left(\frac{M_T^\infty}{M_t^\infty}\right) \geq \mathbb{E}[\log \frac{V_T^\infty}{V_t^\infty}]$.

(ii) $L_t\left(\frac{M_T^\infty}{M_t^\infty}\right) \geq \mathbb{E}_t[\log \frac{V_T^\infty}{V_t^\infty}]$.

(iii) If $\mathbb{E}\left[\log \frac{P_T^T V_T^T}{V_t^T}\right] + L(P_T^T) > 0$, then

$$L\left(\frac{M_T^\infty}{M_t^\infty}\right) \geq \min \left\{ 1, \frac{\mathbb{E}\left[\log \frac{V_T^\infty}{V_t^\infty}\right]}{\mathbb{E}\left[\log \frac{P_T^T V_T^T}{V_t^T}\right]} \right\}.$$ 

(iv) If $\mathbb{E}_t[\log \frac{P_T^T V_T^T}{V_t^T}] > 0$, then

$$L_t\left(\frac{M_T^\infty}{M_t^\infty}\right) \geq \min \left\{ 1, \frac{\mathbb{E}_t[\log \frac{V_T^\infty}{V_t^\infty}]}{\mathbb{E}_t[\log \frac{P_T^T V_T^T}{V_t^T}]} \right\}.$$ 

The proof is based on the martingale property and Jensen’s inequality and is given in Appendix B. The right hand side of the Alvarez-Jermann bound is the expected excess log-return over the time period $[t, T]$ earned on the self-financing portfolio with the wealth process $V$ relative to the return earned from holding the long bond $B^\infty$ over this time period.

6 Long-Term Factorization in Heath-Jarrow-Morton Models

The classical Heath et al. (1992) framework assumes that the family of zero-coupon bond processes $\{(P_t^T)_{t \in [0,T]}, T \geq 0\}$ is sufficiently smooth across the maturity parameter $T$ so that there exists a family of instantaneous forward rate processes $\{(f(t,T))_{t \in [0,T], T \geq 0}\}$ such that

$$P_t^T = e^{-\int_t^T f(t,s)ds},$$

and for each $T$ the forward rate is assumed to follow an Itô process

$$df(t,T) = \mu(t, T)dt + \sigma(t, T) \cdot dW_t^P,$$ 

(6.1)
driven by an $n$-dimensional Brownian motion $W^P = \{(W_t^{P,j})_{t \geq 0}, j = 1,\ldots,n\}$. For each fixed maturity date $T$, the volatility of the forward rate $\sigma(t, T)$ is an $n$-dimensional Itô process. In (6.1) $\cdot$ denotes the Euclidean dot product in $\mathbb{R}^n$. Heath et al. (1992) show that absence of arbitrage implies the drift condition

$$\mu(t, T) = (-\gamma(t) + \sigma^P(t,T)) \cdot \sigma(t, T),$$

where $\gamma(t)$ is the (negative of the) $d$-dimensional market price of Brownian risk $W^P$ and $\sigma^P(t, T) := \int_t^T \sigma(t,u)du$. By Itô formula, zero-coupon bond prices follow Itô processes given by

$$\frac{dP(t,T)}{P(t,T)} = (r(t) + \gamma(t) \cdot \sigma^P(t, T))dt - \sigma^P(t, T) \cdot dW_t^P,$$

where $r(t) = f(t, t)$ is the short rate. Thus, $\sigma^P(t, T)$ defined early as the integral of the forward rate volatility $\sigma(t,u)$ in maturity date $u$ from $t$ to $T$ is identified with the zero-coupon bond volatility. The classical HJM framework treats the forward rate of each maturity as an Itô
where the operator

\[ \text{The drift curve (}) \]

\[ \text{infinite-dimensional (see Prato and Zabczyk (2014) for mathematical foundations of infinite-}
\]

\[ \text{view immediately extends the HJM framework to allow the driving Brownian motion}
\]

\[ \text{By H"older’s inequality, for all}
\]

\[ \text{case (the finite-dimensional case arises by setting}
\]

\[ \text{Chapter 2). We continue to use notation}
\]

\[ \text{sequence of independent standard Brownian motions adapted to (}
\]

\[ \text{H\textsuperscript{∞}}(t, ω, f) \text{take values in the same Hilbert space}
\]

\[ \text{A\textsubscript{f}} \text{in the drift in Eq.(6.2) arises from Musiela’s parameterization,}
\]

\[ \text{A\textsubscript{f}}(x) = \partial_x f(t, ω, f) \text{and is defined more precisely below as the operator in } H_w.
\]

\[ \text{Following Filipovic (2001), we next define the Hilbert space } H_w \text{ of forward curves and give}
\]

\[ \text{conditions on the volatility and drift to ensure that the solution of the HJM evolution equation}
\]

\[ \text{Eq.(6.2) exists in the appropriate sense (the forward curve stays in its prescribed space}
\]

\[ \text{function such that}
\]

\[ \text{infinitely support in a general equilibrium.}
\]

\[ \text{Jin and Glasserman (2001) show how HJM models can be}
\]

\[ \text{et al. (1992) give sufficient conditions on the forward rate volatility and the market price of risk}
\]

\[ \text{process. We thus have a family of Itô processes parameterized by the maturity date. Heath}
\]

\[ \text{An alternative point of view is to treat the forward curve } f \text{ as a single process taking values in an appropriate function space of forward curves. To this end, the Musiela (1993)
\]

\[ \text{parameterization } f(x) := f(t, t + x) \text{ of the forward curve is convenient. Now we work with}
\]

\[ \text{the operator}
\]

\[ \text{df} = (A\text{f} + \mu\text{f}) dt + \sigma\text{f} \cdot dW^P. \quad (6.2)
\]

\[ \text{The infinite-dimensional standard Brownian motion } W^P = \{ (W^P_{t,j})_{t \geq 0, j = 1, 2, ...} \} \text{ is a sequence of independent standard Brownian motions adapted to } (\mathcal{F}_t)_{t \geq 0} \text{ (see Filipovic (2001) Chapter 2). We continue to use notation}
\]

\[ \text{df}(x) = \partial_x f(t, ω, f) \text{ and } A\text{f}(x) \text{ evolves in time) and specifies an arbitrage-free term structure. Let}
\]

\[ \text{the process (}) \]

\[ \text{parametrized by the maturity date. Heath}
\]

\[ \text{An alternative point of view is to treat the forward curve } f \text{ as a single process taking values in an appropriate function space of forward curves. To this end, the Musiela (1993)
\]

\[ \text{parameterization } f(x) := f(t, t + x) \text{ of the forward curve is convenient. Now we work with}
\]

\[ \text{the drift in Eq.(6.2) arises from Musiela’s parameterization,}
\]

\[ \text{H\textsuperscript{∞}(t, ω, f) \text{ take values in the same Hilbert space}
\]

\[ \text{A\textsubscript{f}} \text{ in the drift in Eq.(6.2) arises from Musiela’s parameterization,}
\]

\[ \text{where the operator } A \text{ is interpreted as the first derivative with respect to time to maturity,}
\]

\[ \text{A\textsubscript{f}}(x) = \partial_x f(t, ω, f) \text{ and is defined more precisely below as the operator in } H_w.
\]

\[ \text{Following Filipovic (2001), we next define the Hilbert space } H_w \text{ of forward curves and give}
\]

\[ \text{conditions on the volatility and drift to ensure that the solution of the HJM evolution equation}
\]

\[ \text{Eq.(6.2) exists in the appropriate sense (the forward curve stays in its prescribed space } H_w \text{ as}
\]

\[ \text{it evolves in time) and specifies an arbitrage-free term structure. Let } w : \mathbb{R}_+ \rightarrow [1, \infty) \text{ be a non-decreasing C}^1 \text{ function such that}
\]

\[ \text{we define}
\]

\[ H_w := \{ h \in L^1_{\text{loc}}(\mathbb{R}_+) \mid \exists h' \in L^1_{\text{loc}}(\mathbb{R}_+) \text{ and } \| h \|_w < \infty \},
\]

\[ \| h \|_w^2 := |h(0)|^2 + \int_{\mathbb{R}_+} |h'(x)|^2 w(x) dx.
\]

\[ H_w \text{ is defined as the space of locally integrable functions, with locally integrable (weak) derivatives, and with the finite norm } \| h \|_w. \text{ The finiteness of this norm imposes tail decay on the derivative } h' \text{ of the function (forward curve) in time to maturity such that it decays to zero as time to maturity tends to infinity fast enough so the derivative is square integrable with the weight function } w, \text{ which is assumed to grow fast enough so that the integral}
\]

\[ \text{By Hölder’s inequality, for all } h \in H_w
\]

\[ \int_{\mathbb{R}_+} |h'(x)| dx \leq \left( \int_{\mathbb{R}_+} |h'(x)|^2 w(x) dx \right)^{1/2} \left( \int_{\mathbb{R}_+} w^{-1}(x) dx \right)^{1/2} < \infty.
\]
Thus, \( h(x) \) converges to the limit \( h(\infty) \in \mathbb{R} \) as \( x \to \infty \), which can be interpreted as the long forward rate. In other words, all forward curves in \( H_w \) flatten out at asymptotically long maturities. This is an instance of the theorem due to Dybvig et al. (1996) (see Brody and Hughston (2013) for a recent extension, proof, and related references). We note that existence of the long forward rate is not in itself sufficient for existence of the long bond. The sufficient condition will be given below. The space \( H_w \) equipped with \( \|h\|_w \) is a separable Hilbert space. Define a semigroup of translation operators on \( H_w \) by \((T_t f)(x) = f(t + x)\). By Filipovic (2001) Theorem 5.1.1, it is strongly continuous in \( H_w, h \in H_w \). This is the operator that appears in the drift in Eq. (6.2) due to the Musiela re-parameterization. (The reader can continue to think of it as the derivative of the forward rate in time to maturity.)

We next give conditions on the drift and volatility. First we need to introduce some notations. Define the subspace \( H_w^0 \subset H_w \) by

\[
H_w^0 = \{ f \in H_w \text{ such that } f(\infty) = 0 \}.
\]

For any continuous function \( f \) on \( \mathbb{R}_+ \), define the continuous function \( S f : \mathbb{R}_+ \to \mathbb{R} \) by

\[
(S f)(x) := f(x) \int_0^x f(\eta)d\eta, \quad x \in \mathbb{R}_+.
\]

This operator is used to conveniently express the celebrated HJM arbitrage-free drift condition. By Filipovic (2001) Theorem 5.1.1, there exists a constant \( K \) such that \( \|S h\|_w \leq K \|h\|_w^2 \) for all \( h \in H_w, h \in H_w^0 \). Local Lipschitz property of \( S \) is proved in Filipovic (2001) Corollary 5.1.2, which is used to ensure existence and uniqueness of solution to the HJM equation. Namely, \( S \) maps \( H_w^0 \) to \( H_w^0 \) and is locally Lipschitz continuous:

\[
\|S g - S h\|_w \leq C(\|g\|_w + \|h\|_w)\|g - h\|_w, \forall g, h \in H_w^0,
\]

where the constant \( C \) only depends on \( w \).

Consider \( \ell^2 \), the Hilbert space of square-summable sequences, \( \ell^2 = \{ v = (v_j)_{j \in \mathbb{N}} \in \mathbb{R}^\mathbb{N} \|v\|_2^2 := \sum_{j \in \mathbb{N}} |v_j|^2 < \infty \} \). Let \( e_j \) denote the standard orthonormal basis in \( \ell^2 \). For a separable Hilbert space \( H \), let \( L_0^2(H) \) denote the space of Hilbert-Schmidt operators from \( \ell^2 \) to \( H \) with the Hilbert-Schmidt norm

\[
\|\phi\|_{L_0^2(H)}^2 := \sum_{j \in \mathbb{N}} \|\phi_j\|_H^2 < \infty,
\]

where \( \phi_j := \phi e_j \). We shall identify the operator \( \phi \) with its \( H \)-valued coefficients \( (\phi_j)_{j \in \mathbb{N}} \).

We are now ready to give conditions on the market price of risk and volatility. Recall that we have a filtered probability space \((\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, \mathbb{P})\). Let \( \mathcal{P} \) be the predictable sigma-field. For any metric space \( G \), we denote by \( \mathcal{B}(G) \) the Borel sigma-field of \( G \).

**Assumption 6.1. (Conditions on the Parameters and the Initial Forward Curve)**

(i) The initial forward curve \( f_0 \in H_w \).

(ii) The (negative of the) market price of risk \( \gamma \) is a measurable function from \((\mathbb{R}_+ \times \Omega \times H_w, \mathcal{P} \otimes \mathcal{B}(H_w))\) into \((\ell^2, \mathcal{B}(\ell^2))\) such that there exists a function \( \Gamma \in L^2(\mathbb{R}_+) \) that satisfies

\[
\|\gamma(t, w, h)\|_{\ell^2} \leq \Gamma(t) \quad \text{for all } (t, w, h).
\]

(iii) The volatility \( \sigma = (\sigma^j)_{j \in \mathbb{N}} \) is a measurable function from \((\mathbb{R}_+ \times \Omega \times H_w, \mathcal{P} \otimes \mathcal{B}(H_w))\) into \((L_0^2(H_w), \mathcal{B}(L_0^2(H_w)))\). It is is assumed to be Lipschitz continuous in \( h \) and uniformly bounded, i.e. there exist constants \( D_1, D_2 \) such that for all \( (t, w) \in \mathbb{R}_+ \times \Omega \) and \( h, h_1, h_2 \in H_w \)

\[
\|\sigma(t, w, h) - \sigma(t, w, h_2)\|_{L_0^2(H_w)} \leq D_1 \|h_1 - h_2\|_{H_w}, \quad \|\sigma(t, w, h)\|_{L_0^2(H_w)} \leq D_2.
\]
In the case when $W^P$ is finite-dimensional, simply replace $\ell^2$ with $\mathbb{R}^n$. Filipovic (2001) Remark 2.4.3 states that the Lipschitz constant $D_1$ may depend on $T$, i.e. (6.5) holds for all $(t, \omega) \in [0, T] \times \Omega$.

The drift $\mu_t = \mu(t, \omega, f_t)$ in (6.2) is defined by the HJM drift condition:

$$
\mu(t, \omega, f_t) = \alpha^\text{HJM}(t, \omega, f_t) - (\gamma \cdot \sigma)(t, \omega, f_t),
$$

where $\alpha^\text{HJM}(t, \omega, f_t) = \sum_{j \in \mathbb{N}} S\sigma^j(t, \omega, f_t)$, where $S$ is the previously defined operator (6.3).

The following theorem summarizes the properties of the HJM model (6.2) in this setting.

**Theorem 6.1. (HJM Model)** (i) Eq. (6.2) has a unique continuous weak solution.

(ii) The pricing kernel has the risk-neutral factorization

$$
S_t = \frac{1}{A_t} M_t
$$

with the implied savings account $A$ given by

$$
A_t = \exp \left( \int_0^t f_s(0) ds \right)
$$

and the martingale

$$
M_t = \exp \left( -\frac{1}{2} \int_0^t \|\gamma_s\|_{\ell^2}^2 ds + \int_0^t \gamma_s \cdot dW^P_s \right)
$$

defining the risk-neutral measure $Q|_{\mathcal{F}_t} = M_t^P|_{\mathcal{F}_t}$. The process

$$
W^Q_t := W^P_t - \int_0^t \gamma_s ds
$$

is an (infinite-dimensional) standard Brownian motion under $Q$.

(iii) The $T$-maturity bond price $P^T_t$ follows the process:

$$
P^T_t = A_t \exp \left( \int_0^t \gamma_s \cdot \sigma^T_s ds - \int_0^t \sigma^T_s \cdot dW^P_s - \frac{1}{2} \int_0^t \|\sigma^T_s\|_{\ell^2} ds \right),
$$

where the volatility of the $T$-maturity bond is

$$
\sigma^T_t = \int_{0}^{T-t} \sigma_t(\tau) d\tau.
$$

For the definition of a weak solution used here see Filipovic (2001) Definition 2.4.1. The proof of Theorem 6.1 follows from the results in Filipovic (2001) and summarized for the readers’ convenience in Appendix C.

We are now ready to formulate the main result of this section.

**Theorem 6.2. (Long-term Factorization in the HJM Model)** Suppose the initial forward curve and the market price of risk satisfy Assumption 6.1 (i) and (ii). Suppose the volatility $\sigma = (\sigma^j)_{j \in \mathbb{N}}$ is a measurable function from $(\mathbb{R}_+ \times \Omega \times H_w, \mathcal{P} \otimes \mathcal{B}(H_w))$ into $(L_0^0(H^0_w), \mathcal{B}(L_0^0(H^0_w)))$ and is Lipschitz continuous in $h$ and uniformly bounded as in 6.1 (ii), where $H^0_w \subseteq H^0_w$ with $w'$ satisfying the estimate

$$
\frac{1}{w'(x)} = O(x^{-(3+\epsilon)}).
$$

(6.6)
Then the following results hold.

(i) Eq. (4.1) in Theorem 4.1 holds in the HJM model and, hence, all results in Theorem 4.1 hold.

(ii) Define

\[ \sigma_t^\infty := \int_0^\infty \sigma_t(u)du. \]  

(6.7)

The long bond follows the process:

\[ B_t^\infty = A_t \exp \left( \int_0^t \gamma_s \cdot \sigma_s^\infty ds - \int_0^t \sigma_s^\infty \cdot dW^P_s - \frac{1}{2} \int_0^t \| \sigma_s^\infty \|^2_{L^2} ds \right) \]  

(6.8)

with volatility \( \sigma_t^\infty \). The pricing kernel admits the long-term factorization:

\[ S_t = \frac{1}{B_t^\infty} M_t^\infty \]

with the martingale

\[ M_t^\infty = \exp \left( -\frac{1}{2} \int_0^t \| \gamma_s - \sigma_s^\infty \|^2_{L^2} ds + \int_0^t (\gamma_s - \sigma_s^\infty) \cdot dW^P_s \right) \]  

(6.9)

The long forward measure \( \mathbb{L} \) is given by \( \frac{d\mathbb{L}}{d\mathbb{P}}|_{\mathcal{F}_t} = M_t^\infty \).

(iii) Under \( \mathbb{L} \)

\[ W_t^L := W_t^P + \int_0^t (\sigma_s^\infty - \gamma_s)ds \]

is an (infinite-dimensional) standard Brownian motion, and the \( \mathbb{L} \)-dynamics of the forward curve is:

\[ df_t = (A_t f_t + \alpha_t^HJM - \sigma_t^\infty \cdot \sigma_t)dt + \sigma_t \cdot dW^L_t. \]

The proof is given in Appendix C. The sufficient condition on the forward curve volatility to ensure existence of the long bond is a slight strengthening of Filipovic’s condition in Assumption 6.1. We require that the weight function \( w' \) in the weighted Sobolev space \( H_w \) where the volatility components take their values satisfies the estimate (6.6), a slight strengthening of Filipovic’s assumption on the weight \( w \) of \( H_w \) where the forward curves evolve. Note that we do not impose this estimate on \( w \) in the space of forward curves \( H_w \), only on the space \( H_w' \) featured in the definition of the forward curve volatility. We note that our sufficient conditions ensure that the volatility of the long bond (6.7) is well defined (see proof in Appendix C).

Theorem 6.2 provides a fully explicit long-term factorization in the generally non-Markovian HJM model driven by an infinite-dimensional Brownian motion and illustrates the power of our general framework.

Example 6.1. Gaussian HJM models

When the forward curve volatility \( \sigma \) is deterministic, the model reduces to the Gaussian HJM model. In this case the conditions on \( \sigma \) in Theorem 6.2 simplify to requiring that \( \sigma(t, \omega, h) = \sigma(t) \in L^0_0(H^0_w) \) and is uniformly bounded for some \( w' \) such that \( H^0_w \subseteq H^0_{w'} \) and \( 1/w'(x) = O(x^{-(3+\epsilon)}) \).

When the volatility \( \sigma \) is also independent of time, the model reduces to the generalized Vasicek model extending the classical one-dimensional Vasicek model with \( \sigma^1 = \sigma_r e^{-\kappa x} \) and \( \sigma^j = 0 \) for \( j \geq 2 \). When \( \kappa > 0 \), the short rate \( r_t = f_t(0) \) follows a mean-reverting Vasicek / OU process with constant volatility \( \sigma_r \) and the rate of mean reversion \( \kappa \). Our conditions on \( \sigma \) are automatically satisfied in this case. In particular, in this case the volatility of the long bond is constant, \( \sigma_t^\infty = \sigma_r/\kappa \).
7 Long-Term Factorization in Models of Social Discounting

Our next example is a non-Markovian model of social discount rates due to Brody and Hughston (2013). These authors put forward an arbitrage-free model of social discount rates that is fully consistent with the semimartingale approach to pricing kernels and, at the same time, features the term structure of zero-coupon bond prices (discount factors) $P^T_t$ that decays as $O(T^{-1})$ (or as more general negative power) as maturity becomes asymptotically long. This avoids the heavy discounting of the welfare of future generations in the distant future present in models with exponentially decaying term structures. The problem of determining appropriate social discount rates is of importance in evaluating policies to combat climate change (see Arrow (1995), Arrow et al. (1996), Weitzman (2001) for discussions). Here we demonstrate that Brody and Hughston (2013) models of social discount rates verify our condition (4.1) and, thus, our Theorem 4.1 holds in this class of models. Moreover, we explicitly construct the long-term factorization in these models.

Let $a(t)$ and $b(t)$ be two deterministic strictly positive continuous function of time such that lim$_{t \to \infty} ta(t) = a$ and lim$_{t \to \infty} tb(t) = b$ for some non-negative constants $a$ and $b$ such that $a + b > 0$ and normalized so that $a(0) + b(0) = 1$ (for our purposes we slightly strengthen the assumptions of Brody and Hughston (2013) by assuming that the limits exists, instead of the limits inferior). Let $(M_t)_{t \geq 0}$ be a positive $\mathbb{P}$-martingale with $M_0 = 1$ and define the pricing kernel by

$$S_t = a(t) + b(t)M_t.$$  

Then it is immediate that

$$P^T_t = \frac{a(T) + b(T)M_t}{a(t) + b(t)M_t}.$$  

From the assumptions on $a(t)$ and $b(t)$ we have

$$\lim_{T \to \infty} (TP^T_t) = \frac{a + bM_t}{a(t) + b(t)M_t}$$

for each fixed $t \geq 0$, which implies that $P^T_t$ is $O(T^{-1})$ and, thus, the continuously compounded exponential long rate defined as $-\lim_{T \to \infty} \frac{1}{T} \ln P^T_t$ vanishes in this model. Nevertheless, the so-called tail-Pareto long rates are finite and non-vanishing (see Brody and Hughston (2013)). In particular, for each $t \geq 0$

$$\lim_{T \to \infty} B^T_t = \lim_{T \to \infty} \frac{P^T_t}{P^T_0} = \frac{a + bM_t}{(a + b)(a(t) + b(t)M_t)} \text{ a.s.}$$

We now verify that $M^T_t = S_tB^T_t$ satisfy Eq.(4.1) with

$$M^\infty_t = \frac{a + bM_t}{a + b}$$

so that

$$B^\infty_t = M^\infty_t/S_t = \frac{a + bM_t}{(a + b)(a(t) + b(t)M_t)}.$$

By direct calculation, we have

$$\left| S_t \frac{P^T_t}{P^T_0} - S_t B^\infty_t \right| = \frac{|(ab(T) - a(T)b)M_t - 1|}{(a(T) + b(T))(a + b)} \leq \frac{|aTb(T) - Ta(T)b|}{(Ta(T) + Tb(T))(a + b)} (M_t + 1).$$
Since \( \lim_{T \to \infty} (aTb(T) - Ta(T)b) = 0 \), \( \lim_{T \to \infty} (Ta(T) + Tb(T)) = a + b \) and \( M_{t+1} \) is integrable under \( \mathbb{P} \), it is clear that
\[
\lim_{T \to \infty} \mathbb{E}_T \left[ S_T \frac{P_T}{P_0} - S_t B_t^\infty \right] = 0.
\]
Thus, (4.1) is satisfied and Theorem 4.1 holds.

We can apply Girsanov theorem for semimartingales to explicitly obtain dynamics of \( M \) under the the long forward measure \( L \) defined by the martingale \( M^\infty \). In particular, assume that the (square brackets) quadratic variation \( [M] \) of \( M \) is \( \mathbb{P} \)-locally integrable. Then we can define the predictable (sharp brackets) quadratic variation \( \langle M \rangle \) so that \( [M] - \langle M \rangle \) is a \( \mathbb{P} \)-local martingale. Then by Girsanov Theorem (cf. Jacod and Shiryaev (2003) p.168 Theorem 3.11) the \( \mathbb{P} \)-martingale becomes \( L \)-semimartingale with the canonical decomposition
\[
M_t = \tilde{M}_t + \int_0^t \frac{b\langle M \rangle_s}{a + bM_s} ds,
\]
where \( \tilde{M}_t = M_t - \int_0^t \frac{b\langle M \rangle_s}{a + bM_s} ds \) is a \( L \)-local martingale and \( \int_0^t \frac{b\langle M \rangle_s}{a + bM_s} ds \) is a predictable process of finite variation (“drift”). This example can also be straightforwardly generalized to pricing kernels of the form \( S_t = a(t) + b(t)M_t + c(t)N_t \) driven by several martingales (see Brody and Hughston (2013)).

**Example 7.1. Rational Lognormal Model of Social Discount Rates**

We now specify the martingale \( M \) to be the Dooleans-Dade exponential martingale of a \( \mathbb{P} \)-Brownian motion, \( M_t = e^{\sigma W_t - \frac{1}{2} \sigma^2 t} \). The corresponding pricing kernel \( S_t = a(t) + b(t)e^{\sigma W_t - \frac{1}{2} \sigma^2 t} \) defines the Flesakerand and Hughston (1996) term structure model (7.1). In this case \( [M]_t = (M)_t = \int_0^t \sigma^2 M_s^2 ds \) and
\[
\tilde{M}_t = M_t - \int_0^t \frac{b\sigma^2 M_s^2}{a + bM_s} ds
\]
is a \( L \)-local martingale.

We can give a more detailed characterization by Girsanov’s theorem. Since \( L |_{\mathcal{F}_t} = \frac{a+bM_t}{a+b} \mathbb{P} |_{\mathcal{F}_t} \),
\[
W^L_{t} = W_{t} - \int_0^t \frac{b\sigma M_s}{a + bM_s} ds
\]
is \( L \)-standard Brownian motion. Thus \( M \) satisfies the SDE
\[
dM_t = \frac{b\sigma^2 M_t^2}{a + bM_t} dt + \sigma M_t dW^L_t,
\]
under \( L \), and \( d\tilde{M}_t = \sigma M_t dW^L_t \). Since \( \mathbb{E}^L [\int_0^t \sigma^2 M_s^2 ds] = \int_0^t \mathbb{E}^P [\sigma^2 M_s^2 \left( \frac{a+bM_s}{a+b} \right)] ds < \infty \), \( \tilde{M}_t \) is a true \( L \)-martingale.

8 Long-Term Factorization in Markovian Economies

8.1 Theory

We now further specify our model by assuming that the underlying filtration is generated by a Markov process \( X \) and the pricing kernel is a positive multiplicative functional of \( X \). More
precisely, the stochastic driver of all economic uncertainty is a conservative Borel right process (BRP) \( X = (\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, (X_t)_{t \geq 0}, (\mathbb{P}_x)_{x \in E}) \). A BRP is a continuous-time, time-homogeneous Markov process taking values in a Borel subset \( E \) of some metric space (so that \( E \) is equipped with a Borel sigma-algebra \( \mathcal{E} \)); in applications we can think of \( E \) as a Borel subset of the Euclidean space \( \mathbb{R}^d \), having right-continuous paths and possessing the strong Markov property (i.e., the Markov property extended to stopping times). The probability measure \( \mathbb{P}_x \) governs the behavior of the process \( (X_t)_{t \geq 0} \) when started from \( x \in E \) at time zero. If the process starts from a probability distribution \( \mu \), the corresponding measure is denoted \( \mathbb{P}_\mu \). A statement concerning \( \omega \in \Omega \) is said to hold \( \mathbb{P} \)-almost surely if it is true \( \mathbb{P}_\mu \)-almost surely for all \( x \in E \). The information filtration \( (\mathcal{F}_t)_{t \geq 0} \) in our model is the filtration generated by \( X \) completed with \( \mathbb{P}_\mu \)-null sets for all initial distributions \( \mu \) of \( X_0 \). It is right continuous and, thus, satisfies the usual hypothesis of stochastic calculus. \( X \) is assumed to be conservative, i.e. \( \mathbb{P}_x(X_t \in E) = 1 \) for each initial \( x \in E \) and all \( t \geq 0 \) (the process does not exit the state space \( E \) in finite time, i.e. no killing or explosion).

Cinlar et al. (1980) show that stochastic calculus of semimartingales defined over a right process can be set up so that all key properties hold simultaneously for all starting points \( x \in E \) and, in fact, for all initial distributions \( \mu \) of \( X_0 \). In particular, an \( (\mathcal{F}_t)_{t \geq 0} \)-adapted process \( S \) is an \( \mathbb{P}_x \)-semimartingale (local martingale, martingale) simultaneously for all \( x \in E \) and, in fact, for all \( \mathbb{P}_\mu \), where \( \mu \) is the initial distribution (of \( X_0 \)). With some abuse of notation, in this section we simply write \( \mathbb{P} \) where, in fact, we are dealing with the family of measures \( (\mathbb{P}_x)_{x \in E} \) indexed by the initial state \( x \). Correspondingly, we simply say that a process is a \( \mathbb{P} \)-semimartingale (local martingale, martingale), meaning that it is a \( \mathbb{P}_x \)-semimartingale (local martingale, martingale) for each \( x \in E \).

The advantage of working in this generality of Borel right processes is that we can treat processes with discrete state spaces (Markov chains), diffusions in the whole Euclidean space or in a domain with a boundary and some boundary behavior, as well as pure jump and jump-diffusion processes in the whole Euclidean space or in a domain with a boundary, all in a unified fashion.

We assume that the pricing kernel \( (S_t)_{t \geq 0} \) is a positive \(( (\mathcal{F}_t)_{t \geq 0}, \mathbb{P} )\)-semimartingale that, in addition to Assumption 2.1, is also a multiplicative functional of \( X \), i.e.

\[
S_{t+s}(\omega) = S_t(\omega)S_s(\theta_t(\omega)),
\]

where \( \theta_t : \Omega \to \Omega \) is the shift operator (i.e. \( X_t(\theta_s(\omega)) = X_{t+s}(\omega) \)). We also assume integrability \( \mathbb{E}_x^\mathbb{P}[S_t] < \infty \) for all \( t \) and each \( x \in E \) so that Assumption 2.1 is satisfied.

Then the time-\( s \) price of a payoff \( f(X_t) \) at time \( t \geq s \geq 0 \) is

\[
\mathcal{P}_{t-s}f(X_s) = \mathbb{E}_x^\mathbb{P}\left[ \frac{S_t}{S_s} f(X_t) \Big| \mathcal{F}_s \right],
\]

where we used the Markov property and time homogeneity and introduced a family of pricing operators \(( \mathcal{P}_t )_{t \geq 0} \):

\[
\mathcal{P}_t f(x) := \mathbb{E}_x^\mathbb{P}[S_t f(X_t)],
\]

where \( \mathbb{E}_x^\mathbb{P} \) denotes the expectation with respect to \( \mathbb{P}_x \). The pricing operator \( \mathcal{P}_t \) maps the payoff function \( f \) at time \( t \) into its present value function at time zero as a function of the initial state \( X_0 = x \).
Suppose $\mathcal{P}_t$ have a positive eigenfunction $\pi$ satisfying

$$
\mathcal{P}_t \pi(x) = e^{-\lambda t} \pi(x)
$$

for some $\lambda \in \mathbb{R}$ and all $t > 0, x \in E$. Then the process

$$
M_t^\pi = S_t e^{\lambda t} \frac{\pi(X_t)}{\pi(X_0)}
$$

is a positive $\mathbb{P}$-martingale. The key observation of Hansen and Scheinkman (2009) is that then the pricing kernel admits a factorization

$$
S_t = e^{-\lambda t} \frac{\pi(X_0)}{\pi(X_t)} M_t^\pi. \tag{8.1}
$$

Since $M_t^\pi$ is a positive $\mathbb{P}$-martingale starting from one, we can define a new probability measure $\mathbb{Q}^\pi$ (eigen-measure) associated with the eigenfunction $\pi$ by:

$$
\mathbb{Q}^\pi | \mathcal{F}_t = M_t^\pi \mathbb{P} | \mathcal{F}_t.
$$

The pricing operator can then be expressed under $\mathbb{Q}^\pi$ as

$$
\mathcal{P}_t f(x) = e^{-\lambda t} \frac{\pi(x)}{\pi(X_t)} \mathbb{E}_{X_t}^{\mathbb{Q}^\pi} \left[ \frac{f(X_t)}{\pi(X_t)} \right].
$$

We refer the reader to Hansen and Scheinkman (2009) and Qin and Linetsky (2014) for more on factorizations of multiplicative pricing kernels.

In general, for a given pricing kernel $S$, there may be no, one, or multiple positive eigenfunctions. However, Qin and Linetsky (2014) prove that there exists at most one positive eigenfunction such that the process is recurrent under $\mathbb{Q}^\pi$ (for the precise definition of a recurrent BRP, see Qin and Linetsky (2014) Definition 2.2). If such an eigenfunction exists, Qin and Linetsky (2014) call it a recurrent eigenfunction, denote it by $\pi_R$, and call the associated $\mathbb{Q}^\pi_R$ recurrent eigen-measure. The corresponding eigenvalue is denoted as $\lambda_R$. Existence of a recurrent eigenfunction is investigated in Qin and Linetsky (2014), where several sufficient conditions are obtained and analytical solutions are provided for a range of specific models.

We now give a sufficient condition that leads to the identification $\mathbb{Q}^\pi_R = \mathbb{L}$.

**Assumption 8.1. (Exponential Ergodicity)** Assume a recurrent eigenfunction $\pi_R$ exists and under the recurrent eigen-measure $\mathbb{Q}^\pi_R$ $X$ satisfies the following exponential ergodicity assumption. There exists a probability measure $\varsigma$ on $E$ and some positive constants $c, \alpha, t_0$ such that the following exponential ergodicity estimate holds for all Borel functions satisfying $|f| \leq 1$:

$$
\left| \mathbb{E}_{X_t}^{\mathbb{Q}^\pi_R} \left[ \frac{f(X_t)}{\pi_R(X_t)} \right] - \int_E \frac{f(y)}{\pi_R(y)} \varsigma(dy) \right| \leq \frac{c}{\pi_R(x)} e^{-\alpha t} \tag{8.2}
$$

for all $t \geq t_0$ and each $x \in E$.

Under this assumption, the distribution of $X$ converges to the limiting distribution $\varsigma$ at the exponential rate $\alpha$. Sufficient conditions for (8.2) for Borel right processes can be found in Theorem 6.1 of Meyn and Tweedie (1993).
Theorem 8.1. (Identification of $\mathbb{L}$ and $Q^{\pi R}$) If Assumption 8.1 is satisfied, then Eq.(4.1) holds with $M^\infty_t = M^{\pi R}_t$ and, thus, Theorem 4.1 applies, the long bond is given by

$$B^\infty_t = e^{\lambda_R t} \frac{\pi_R(X_t)}{\pi_R(X_0)},$$

and the recurrent eigen-measure coincides with the long forward measure:

$$Q^{\pi R} = \mathbb{L}.$$ 

The proof is given in Appendix D. This result supplies a sufficient condition for existence of the long forward measure and the long-term factorization and is close to the results in Hansen and Scheinkman (2009), but is distinct from them in several respects. Here our development starts with defining the long bond and the long forward measure. There is no issue of uniqueness by definition, but there is an issue of existence. In Markovian economies, we show that if the pricing kernel possesses a recurrent eigenfunction $\pi_R$ and if the Markov process moreover satisfies the exponential ergodicity assumption (8.1) under the recurrent eigen-measure $Q^{\pi R}$, then the long forward measure exists and is identified with the recurrent eigen-measure. The pricing kernel then possesses the long-term factorization, and the long-term factorization is identified with the eigen-factorization of Hansen and Scheinkman (2009).

We stress that Assumption 8.1 is merely a sufficient condition for the identification $Q^{\pi R} = \mathbb{L}$. This identification may hold in models not satisfying Assumption 8.1. On the other hand, in general (when Assumption 8.1 is not satisfied), it may be the case that $\mathbb{L}$ exists while $Q^{\pi R}$ does not (this case is illustrated in Section 8.2.4 below), that $Q^{\pi R}$ exists while $\mathbb{L}$ does not, or even that both $\mathbb{L}$ and $Q^{\pi R}$ exist but are distinct.

8.2 Markovian Examples

8.2.1 Hunt Processes with Duals

The example in this section closely follows the class of pricing kernels investigated in Section 5.1 of Qin and Linetsky (2014). In this example we assume that $X$ is a conservative Hunt process on a locally compact separable metric space $E$. This entails making additional assumptions that the Borel right process $X$ on $E$ also has sample paths with left limits and is quasi-left continuous (no jumps at predictable stopping times, and fixed times in particular). In this section we further assume that the pricing kernel admits an absolutely continuous non-decreasing implied savings account (recall Section 3) $A_t = e^{\int_0^t r(X_s) ds}$ with the non-negative short rate function $r(x)$. The risk-neutral measure $Q$ can then be defined, and the pricing operators take the form

$$\mathcal{P}_t f(x) = \mathbb{E}_Q[e^{-\int_0^t r(X_s) ds} f(X_t)]$$

under $Q$. Let $X^r$ denote $X$ killed at the rate $r$ (i.e. the process is killed (sent to an isolated cemetery state) at the first time the positive continuous additive functional $\int_0^t r(X_s) ds$ exceeds an independent unit-mean exponential random variable). It is a Borel standard process (see Definition A.1.23 and Theorem A.1.24 in Chen and Fukushima (2011)) since it shares the sample path with the Hunt process $X$ prior to the killing time. The pricing semigroup $(\mathcal{P}_t)_{t \geq 0}$ is then identified with the transition semigroup of the Borel standard process $X^r$. 

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We further assume that there is a positive sigma-finite reference measure \( m \) with full support on \( E \) such that \( X_t \) has a dual with respect to \( m \). That is, there is a strong Markov process \( X_t \) on \( E \) with the semigroup \( \{ \mathcal{P}_t \}_{t \geq 0} \) such that for any \( t > 0 \) and non-negative functions \( f \) and \( g \):

\[
\int_E f(x) \mathcal{P}_t g(x) m(dx) = \int_E g(x) \hat{\mathcal{P}}_t f(x) m(dx).
\]

We further make the following assumptions.

**Assumption 8.2.** (i) There exists a family of continuous and strictly positive functions \( p(t, \cdot, \cdot) \) on \( E \times E \) such that for any \( t \in (0, \infty) \times E \) and any non-negative function \( f \) on \( E \),

\[
\mathcal{P}_t f(x) = \int_E p(t, x, y) f(y) m(dy), \quad \hat{\mathcal{P}}_t f(x) = \int_E p(t, y, x) f(y) m(dy).
\]

(ii) The density satisfies:

\[
\int_E \int_E p^2(t, x, y) m(dx) m(dy) < \infty, \quad \forall t > 0.
\]

(iii) There exists some \( T > 0 \) such that

\[
\sup_{x \in E} \int_E p^2(t, x, y) m(dy) < \infty, \quad \sup_{x \in E} \int_E p^2(t, y, x) m(dy) < \infty, \quad \forall t \geq T.
\]

Under these assumptions, Qin and Linetsky (2014) prove the following results.

**Theorem 8.2.** Suppose Assumption 8.2 is satisfied. Then the following results hold.

(i) The pricing operator \( \mathcal{P}_t \) and the dual operator \( \hat{\mathcal{P}}_t \) possess unique positive, continuous, bounded eigenfunctions \( \pi_R(x) \) and \( \hat{\pi}_R(x) \) belonging to \( L^2(E, m) \):

\[
\int_E p(t, x, y) \pi_R(y) m(dy) = e^{-\lambda_R t} \pi_R(x), \quad \int_E p(t, y, x) \hat{\pi}_R(y) m(dy) = e^{-\lambda_R t} \hat{\pi}_R(x)
\]

with some \( \lambda_R \geq 0 \) for each \( t > 0 \) and every \( x \in E \).

(ii) Let \( C := \int_E \pi_R(x) \hat{\pi}_R(x) m(dx) \). There exist constants \( b, \gamma > 0 \) and \( T' > 0 \) such that for \( t \geq T' \) we have the estimate for the density

\[
|Ce^{\lambda_R t} p(t, x, y) - \pi_R(x) \hat{\pi}_R(y)| \leq be^{-\gamma t}, \quad x, y \in E.
\]

(iii) The process \( X \) is recurrent under \( \mathbb{Q}^x \) defined by \( \mathbb{Q}^x \|_{\mathcal{F}_t} = \hat{M}^{x, t} \mathbb{Q} \|_{\mathcal{F}_t} \) with

\[
\hat{M}^{x, t} = e^{-\int_0^t \tau(X_s) ds + \lambda_R \frac{\pi_R(X_t)}{\pi_R(X_0)}}
\]

for each \( x \in E \). Moreover, \( X \) is positive recurrent under \( \mathbb{Q}^x \) with the stationary distribution \( \mu(dx) = C^{-1} \pi_R(x) \hat{\pi}_R(x) m(dx) \).

(iv) If in addition \( m \) is a finite measure, \( m(E) < \infty \), then the zero-coupon bond has the following long maturity estimate:

\[
P(x, t) = \int_E p(t, x, y) m(dy) = c_0 \pi_R(x) e^{-\lambda_R t} + O(e^{-(\gamma + \lambda_R) t})
\]

with \( c_0 = \int_E (1/\pi_R(x)) \mu(dx) = C^{-1} \int_E \hat{\pi}_R(x) m(dx) \) for each \( x \in E \).
The proof is given in Qin and Linetsky (2014) and is based on Zhang et al. (2013) which, in turn, is based on Jentzsch’s theorem, a counterpart of Perron-Frobenius theorem for integral operators in $L^2$ spaces. (A further extension of Jentzsch’s theorem to operators in abstract Banach spaces is provided by the Krein-Rutman theorem).

In the special case when $\mathcal{P}_t = \hat{\mathcal{P}}_t$, i.e. the pricing operators are symmetric with respect to the measure $m$, $(\mathcal{P}_t)_{t \geq 0}$ can be interpreted as the transition semigroup of a symmetric Markov process $X^\tau$ killed at the rate $r$ (cf. Chen and Fukushima (2011) and Fukushima et al. (2010)). In particular, essentially all one-dimensional diffusions are symmetric Markov processes with the speed measure $m$ acting as the symmetry measure. In the symmetric case, Assumption 8.2 (ii) implies that for each $t > 0$ the pricing operator $\mathcal{P}_t$ is a symmetric Hilbert-Schmidt operator in $L^2(E, m)$. It further implies that the pricing semigroup is trace class (cf. Davies (2007) Section 7.2) and, hence, for each $t > 0$ the pricing operator $\mathcal{P}_t$ has a purely discrete spectrum \{\(e^{-\lambda_n t}, n = 1, 2, \ldots\)\} with $0 \leq \lambda_1 \leq \lambda_2 \leq \ldots$ repeated according to the eigenvalue multiplicity with the finite trace

$$
\text{tr} \mathcal{P}_t = \int_E p(t, x, x) m(dx) = \sum_{n=1}^{\infty} e^{-\lambda_n t} < \infty.
$$

Using the symmetry of the density, $p(t, x, y) = p(t, y, x)$, and the Chapman-Kolmogorov equation, Assumption 8.2 (iii) reduces to the assumption that there exists a constant $T > 0$ such that

$$
\sup_{x \in E} p(t, x, x) < \infty, \; \forall t \geq T.
$$

Under the recurrent eigen-measure $Q^{\pi_R}$ ensured to exist under assumptions of Theorem 8.2, the transition function of $X$ is given by

$$
Q^{\pi_R}(t, x, dy) = e^{\lambda_R t} \frac{\pi_R(y)}{\pi_R(x)} p(t, x, y) m(dy)
$$

and possesses the stationary measure

$$
\varsigma(dy) = C^{-1} \pi_R(y) \hat{\pi}_R(y) m(dy).
$$

**Theorem 8.3.** In addition to assumptions in Theorem 8.2 assume that $m$ is a finite measure, i.e. $m(E) < \infty$. Then the long-term measure exists and is identified with the recurrent eigen-measure, i.e. $\mathbb{L} = Q^{\pi_R}$.

The proof is by verifying Assumption 8.1 and, hence, establishing that in this case Theorem 8.1 applies. For any function $f$ such that $|f| \leq 1$ and $t \geq t_0$ we have

$$
\left| E^{Q^{\pi_R}} \left[ \frac{f(X_t)}{\pi_R(X_t)} \right] - \int_E \frac{f(y)}{\pi_R(y)} \varsigma(dy) \right| = \left| \int_E (Q^{\pi_R}(t, x, dy) - \varsigma(dy)) \frac{f(y)}{\pi_R(y)} \right|
$$

$$
\leq \int_E |Q^{\pi_R}(t, x, dy) - \varsigma(dy)| \frac{1}{\pi_R(y)}
$$

$$
= \int_E C^{-1} \frac{\pi_R(y)}{\pi_R(x)} m(dy) |C e^{\lambda_R t} p(t, x, y) - \pi_R(x) \hat{\pi}_R(y)| \frac{1}{\pi_R(y)}
$$

$$
\leq \frac{1}{\pi_R(x)} \int_E C^{-1} m(dy) b e^{-\gamma t}.
$$

This verifies (8.2) with $\alpha = \gamma$ and $c = C^{-1} b \int_E m(dy)$. We stress that assuming $m$ is finite also ensure $P^T_t = \mathcal{P}_{T-t}1(X_t) < \infty$ for all $T > t \geq 0$ and thus consistent with our general Assumption 2.1.

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8.2.2 Cox-Ingersoll-Ross Model

Consider a CIR diffusion
\[ dX_t = (a - \kappa X_t)dt + \sigma \sqrt{X_t}dW_t^Q, \quad (8.3) \]
where \( a > 0, \sigma > 0 \) and \( \kappa \in \mathbb{R} \). In the CIR model the short rate is taken to be \( r_t = X_t \). We interpret (8.3) as the risk-neutral dynamics. The closed form expression for the density \( p(t, x, y) \) of the pricing kernel \( \mathcal{P}_t \) is available (cf. Cox et al. (1985)). Qin and Linetsky (2014) (Section 6.1.1) verify that it satisfies all assumptions in Theorem 8.3 and thus the long forward measure \( \mathbb{L} \) exists in the CIR model and coincides with the recurrent eigen-measure \( \mathbb{Q}^{\pi_R} \). Qin and Linetsky (2014) determine the recurrent eigenfunction
\[ \pi_R(x) = e^{-\frac{\gamma}{\sigma^2}x} \]
and the corresponding eigenvalue
\[ \lambda_R = \frac{a}{\sigma^2}(\gamma - \kappa), \]
where \( \gamma = \sqrt{\kappa^2 + 2\sigma^2} \). From their results, the long bond in the CIR model is:
\[ B_t^\infty = e^{\lambda_R t} \frac{\pi_R(X_t)}{\pi_R(X_0)} = e^{\frac{\gamma - \kappa}{\sigma^2}(X_t - X_0)}. \]
(This expression can also be obtained directly by taking the limit \( T \to \infty \) of the ratio \( P(X_t, T - t)/P(X_0, T) \), where \( P(x, \tau) \) is the CIR bond pricing function.) The \( \mathbb{L} \)-measure dynamics of \( X \) can be obtained from Girsanov’s theorem by using the \( \mathbb{Q} \)-martingale \( B_t^\infty/A_t = e^{-\int_0^t X_s ds}B_t^\infty \) to change measure from \( \mathbb{Q} \) to \( \mathbb{L} \):
\[ dX_t = (a - \gamma X_t)dt + \sigma \sqrt{X_t}dW_t^\mathbb{L}. \]
We observe that the rate of mean reversion under \( \mathbb{L} \) is higher than under \( \mathbb{Q} \): \( \gamma = \sqrt{\kappa^2 + 2\sigma^2} > \kappa \).

8.2.3 Vasicek Model

Consider an Ornstein-Uhlenbeck (OU) process
\[ dX_t = \kappa(\theta - X_t)dt + \sigma dW_t^Q \quad (8.4) \]
with \( \kappa > 0, \sigma > 0 \). In the Vasicek model the short rate is taken to be \( r_t = X_t \). We interpret (8.4) as the risk-neutral dynamics. Qin and Linetsky (2014) determine the recurrent eigenfunction
\[ \pi_R(x) = e^{-\frac{\theta}{\kappa^2}x} \]
and the corresponding eigenvalue \( \lambda_R = \theta - \frac{\sigma^2}{2\kappa^2} \) in the Vasicek model, and obtain the \( \mathbb{Q}^{\pi_R} \)-dynamics:
\[ dX_t = \kappa(\theta - \frac{\sigma^2}{\kappa^2} - X_t)dt + \sigma dW_t^{\mathbb{Q}^{\pi_R}}. \quad (8.5) \]
In contrast to CIR, in the Vasicek model the short rate is not non-negative, and we cannot apply Theorem 8.3. Nevertheless, in Appendix E.1 we directly verify that the \( L^1 \) convergence condition in Theorem 4.1 holds in Vasicek model and, hence, the long bond exists and is given by
\[ B_t^\infty = e^{\lambda_R t} \frac{\pi_R(X_t)}{\pi_R(X_0)} = e^{(\theta - \frac{\sigma^2}{2\kappa^2} - \frac{1}{\kappa^2}X_t)(X_t - X_0)}, \]
and, thus, the long forward measure \( \mathbb{L} \) can be identified with \( \mathbb{Q}^{\pi_R} \) in the Vasicek model. Thus, we identify the \( \mathbb{L} \)-dynamics with (8.5). We note that the long-run mean \( \theta - \frac{\sigma^2}{\kappa^2} < \theta \) is lower under \( \mathbb{L} \) than under \( \mathbb{Q} \) in the Vasicek model.
8.2.4 A Square-root Model with Absorption at Zero: \( L \) Exists, \( Q^\pi \) Does Not Exist

Consider a degenerate CIR model

\[
dX_t = -\kappa X_t dt + \sigma \sqrt{X_t} dW^Q_t,
\]

where \( \sigma > 0 \) and \( \kappa \in \mathbb{R} \). The parameter \( \alpha \) in the CIR SDE of Section 8.2.2 now vanishes, and the process has an absorbing boundary at zero. The short rate is taken to be \( r_t = X_t \). We interpret (8.6) as the risk-neutral dynamics. It is clear that under any locally equivalent measure the boundary at zero remains absorbing and, thus, no recurrent eigenfunction exists. However, Appendix E.2 shows that Theorem 4.1 nevertheless applies, and the long bond is given by:

\[
B^\infty_t = e^{-\frac{\kappa}{\sigma^2}(X_t - X_0)},
\]

where \( \gamma = \sqrt{\kappa^2 + 2\sigma^2} \). The \( L \)-measure dynamics of \( X \) can then be immediately obtained from Girsanov’s theorem by using the \( Q \)-martingale \( B^\infty_t / A_t = e^{-\int_0^t X_s ds} B^\infty_t \) to change from \( Q \) to \( L \):

\[
dX_t = -\gamma X_t dt + \sigma \sqrt{X_t} dW^L_t
\]

with \( \gamma = \sqrt{\kappa^2 + 2\sigma^2} \). This example is interesting as it illustrates existence of the long bond, long forward measure and, hence, the long-term factorization in a transient case. In this case the recurrent eigen-measure \( Q^\pi \) does not exist, while the long forward measure \( L \) nevertheless exists, but \( X \) is transient under \( L \).

9 A Semimartingale Perspective on Ross’ Recovery

We conclude with an application to the Recovery Theorem of Ross (2013). Ross asks an interesting question: under what assumptions can one uniquely recover the market’s beliefs about physical probabilities from Arrow-Debreu state prices implied by observed market prices of derivative securities, such as options? Ross’ Recovery Theorem provides the following answer. If all uncertainty in an arbitrage-free, frictionless economy follows a finite state, discrete time irreducible Markov chain and the pricing kernel satisfies a structural assumption of transition independence, then there exists a unique recovery of physical probabilities from given Arrow-Debreu state prices. Ross’ proof crucially relies on the Perron-Frobenius theorem establishing existence and uniqueness of a positive eigenvector of an irreducible non-negative matrix. Carr and Yu (2012) observe that Ross’ recovery result can be extended to 1D diffusions on bounded intervals with regular boundaries at both ends by observing that the infinitesimal generator of such a diffusion is a regular Sturm-Liouville operator that has a unique positive eigenfunction. They then rely on the regular Sturm-Liouville theory to show uniqueness of Ross recovery in that setting.

Hansen and Scheinkman (2014) and Borovička et al. (2014) point out that Ross’ assumption of transition independence of the Markovian pricing kernel is a specialization of the factorization in Hansen and Scheinkman (2009) to the case where the martingale component is degenerate, \( M^\pi = 1 \). Qin and Linetsky (2014) investigate general continuous-time Markovian environments with all uncertainty driven by a Borel right process and also show that transition independence restricts the pricing kernel to the form (8.1) for some positive eigenfunction \( \pi(x) \) with the corresponding eigenvalue \( \lambda \) and with the degenerate martingale \( M^\pi = 1 \). Borovička et al.
(2014) further point out that under ergodicity assumptions that both fix uniqueness of a positive eigenfunction and identify the corresponding eigen-measure $Q^\pi$ with the measure we call in the present paper the long-term forward measure $\mathbb{L}$, the transition independence assumption leads to the identification of the physical measure $P$ with the long forward measure $\mathbb{L}$. Martin and Ross (2013), working in discrete time, ergodic finite-state Markov chain environments, present similar results.

This previous literature crucially relies on the Markov property. We now discuss Ross’ Recovery Theorem from the semimartingale perspective of this paper that does not require the Markov assumption. Since the economic implication of transition independence of the pricing kernel in Ross’ discrete-time, finite state irreducible Markov chain economy is the identification $M^\pi = 1$ and, consequently, the identification $P = \mathbb{L}$, from the perspective of this paper Ross’ assumption can be directly extended to semimartingale models by restricting the pricing kernel to what we called long-term risk-neutral in Definition 4.3 of Section 4. In other words, Ross’ pricing kernel discounts at the rate of return on the long bond, assumes away the martingale $M^\infty_t = 1$, and, hence, identifies $P = \mathbb{L}$. By Proposition 4.1, an immediate economic consequence of this restriction on the pricing kernel is the growth optimality of the long bond. In particular, we can immediately extend Ross’ recovery to HJM models by setting the martingale (6.9) to unity and, thus, identifying the market price of risk in the HJM model with the volatility of the long bond:

$$\gamma_t = \sigma^\infty_t.$$  

In other words, if we start with a given risk-neutral HJM model under $Q$, as long as the HJM volatility structure satisfies sufficient conditions in Section 6, Theorem 6.2 explicitly identifies the long forward measure $\mathbb{L}$. Then Ross’ recovery further identifies $P = \mathbb{L}$. We thus extend Ross’ recovery and Borovička et al. (2014) interpretation of it to semimartingale environments without assuming the Markov property.

We point out that Alvarez-Jermann and Bakshi-Chabi-Yo bounds on the components of the long-term factorization (extended to semimartingale environments in Theorems 5.2 and 5.3) allow direct empirical testing of the magnitude of the volatility of the martingale component $M^\infty_t$ in the long term factorization. In particular, they give estimates of the magnitude of the specification error resulting from assuming away the martingale $M^\infty_t = 1$. Borovička et al. (2014) point out that the empirical results of Alvarez and Jermann (2005) and Bakshi and Chabi-Yo (2012) imply that the martingale component is highly non-trivial, and neglecting it would lead to substantial model misspecification. Further empirical investigations of specification errors resulting from the identification $P = \mathbb{L}$ could address the issues of selecting an appropriate empirically observable proxy for the long bond (thirty year maturity of US Treasuries may not be long enough to provide a sufficiently good proxy for the long bond) that arises in testing Ross’ recovery already in the setting of bond market returns and, further, a possible “survivorship bias” of the high U.S. equity premiums observed in the 1900s to the 2000s sample.

10 Conclusion

This paper extends the long-term factorization of the pricing kernel due to Alvarez and Jermann (2005) in discrete-time ergodic environments and Hansen and Scheinkman (2009) in continuous ergodic Markovian environments to general semimartingale environments without assuming Markov property. We show how the long bond and the long forward measure can be defined
in semimartingale environments, extend Alvarez and Jermann (2005) and Bakshi and Chabi-Yo (2012) bounds, obtain explicit long-term factorizations in (generally non-Markovian) Heath-Jarrow-Morton models and models of social discounting, provide sufficient conditions for existence of the long-term factorization in Markovian models, complementing the results of Hansen and Scheinkman (2009), and extend to general semimartingale environments the interpretation of Ross (2013) recovery by Borovička et al. (2014).

A Proof of Theorem 4.1

We first recall some results about semimartingale convergence originally introduced by Émery (1979) (see Czichowsky and Schweizer (2011), Kardaras (2013) and Cuchiero and Teichmann (2014) for recent applications of Emery’s semimartingale topology in mathematical finance). Emery’s distance between two semimartingales $X$ and $Y$ is defined by

$$d_S(X, Y) = \sum_{n \geq 1} 2^{-n} \sup_{|\eta| \leq 1} \left( \mathbb{E}[1 \wedge |\eta_0(X_0 - Y_0) + \int_0^n \eta_s d(X - Y)_s]|]\right),$$  

(A.1)

where the supremum is taken over all predictable processes $\eta$ bounded by one, $|\eta_t| \leq 1$ and $\int_0^t \eta_s dX_s$ denotes the stochastic integral of a predictable process $\eta$ with respect to a semimartingale $X$ (cf. Jacod and Shiryaev (2003)). When we wish to emphasize the probability measure, we write $d_P S(X, Y)$. With this metric, the space of semimartingales is a complete topological vector space, and the corresponding topology is called the semimartingale topology. We denote convergence in the semimartingale topology by $S \rightarrow$. By Theorem II.5 in Memin (1980), the semimartingale convergence is invariant under locally equivalent measure transformations.

The semimartingale topology is stronger than the topology of uniform convergence in probability on compacts (ucp), for which $\eta$ only takes the form $\eta(x) = 1_{[0,s]}(x)$ in (A.1):

$$d_{ucp}(X, Y) = \sum_{n \geq 1} 2^{-n} \mathbb{E}[1 \wedge \sup_{s \leq n} |X_s - Y_s|].$$

The following inequality due to Burkholder is useful for proving convergence in the semimartingale topology in Theorem 4.1 (see Meyer (1972) Theorem 47, p.50 for discrete martingales and Cuchiero and Teichmann (2014) for continuous martingales, where a proof is provided inside the proof of their Lemma 4.7).

**Lemma A.1.** For every martingale $X$ and every predictable process $\eta$ bounded by one, $|\eta_t| \leq 1$, it holds that

$$a \mathbb{P}( \sup_{s \in [0,t]} |\int_0^s \eta_s dX_u| > a) \leq 18 \mathbb{E}^\mathbb{P}[|X_t|]$$

for all $a \geq 0$ and $t > 0$.

We will also use the following result (cf. Kardaras (2013), Proposition 2.10).

**Lemma A.2.** If $X^n \Rightarrow X$ and $Y^n \Rightarrow Y$, then $X^n Y^n \Rightarrow XY$.

We will also make use of the following lemma.

**Lemma A.3.** If $(X^n_t)_{t \geq 0}$ is a family of martingales and there exists a process $(X^\infty_t)_{t \geq 0}$ such that for each $t$, $X^n_t \rightarrow_{L^1} X^\infty_t$ as $n \rightarrow \infty$, then $(X^\infty_t)_{t \geq 0}$ is a martingale.
Proof. By Jensen’s inequality, for \( t > s \),
\[
\mathbb{E}[X_t^n - X_t^\infty|\mathcal{F}_s] \leq \mathbb{E}[|X_t^n - X_t^\infty||\mathcal{F}_s].
\]
Thus, for each \( t \) and \( s \) such that \( t > s \)
\[
\mathbb{E}[|X_s^n - \mathbb{E}[X_t^\infty|\mathcal{F}_s]|] = \mathbb{E}\left[\mathbb{E}[|X_t^n - X_t^\infty|\mathcal{F}_s]|\right] \leq \mathbb{E}\left[\mathbb{E}[|X_t^n - X_t^\infty||\mathcal{F}_s]\right] = \mathbb{E}[X_t^n - X_t^\infty].
\]
Since \( X_t^n \overset{L^1}{\rightarrow} X_t^\infty \) as \( n \to \infty \), above inequality implies that \( X_s^n \overset{L^1}{\rightarrow} \mathbb{E}[X_t^\infty|\mathcal{F}_s] \) as \( n \to \infty \). On the other hand, it also holds that \( X_s^n \overset{L^1}{\rightarrow} X_s^\infty \) as \( n \to \infty \). Thus \( X_s^\infty = \mathbb{E}[X_t^\infty|\mathcal{F}_s] \). This verifies \((X_t^\infty)_{t \geq 0}\) is a martingale. □

We are now ready to prove Theorem 4.1.

(i) Since \((M_t^T)_{t \geq 0}\) are positive \(\mathbb{P}\)-martingales with \(M_0^T = 1\), and for each \( t \geq 0 \) random variables \( M_t^T \) converge to \( M_t^\infty = 0 \) in \( L^1 \), by Lemma A.3 \((M_t^\infty)_{t \geq 0}\) is also a positive \(\mathbb{P}\)-martingale with \(M_0^\infty = 1\). Emery’s distance between the martingales \(M^T\) for some \( T > 0 \) and \( M^\infty \) is

\[
d_M^t(M^T, M^\infty) = \sum_{n \geq 1} 2^{-n} \sup_{|n| \leq 1} \mathbb{E}^t[1 \land |\int_0^n \eta_s d(M^T - M^\infty)|].
\]

To prove \(M^T \overset{\mathcal{S}}{\rightarrow} M^\infty\), it is suffice to prove that for all \( n \)
\[
\lim_{T \to \infty} \sup_{|n| \leq 1} \mathbb{E}^t[1 \land |\int_0^n \eta_s d(M^T - M^\infty)|] = 0. \tag{A.2}
\]

We can write for an arbitrary \( \epsilon > 0 \) (for any random variable \( X \) it holds that \( \mathbb{E}[1 \land |X|] \leq \mathbb{P}(|X| > \epsilon) + \epsilon \))
\[
\mathbb{E}^t[1 \land |\int_0^n \eta_s d(M^T - M^\infty)|] \leq \mathbb{P}(\int_0^n \eta_s d(M^T - M^\infty)| > \epsilon) + \epsilon.
\]

By Lemma A.1,
\[
\sup_{|n| \leq 1} \mathbb{E}^t[1 \land |\int_0^n \eta_s d(M^T - M^\infty)|] \leq \frac{18}{\epsilon} \mathbb{E}^t[|M_n^T - M_n^\infty|] + \epsilon.
\]

Since \(\lim_{T \to \infty} \mathbb{E}^t[|M_n^T - M_n^\infty|] = 0\), and \( \epsilon \) can be taken arbitrarily small, Eq.(A.2) is verified and, hence, \(M^T \overset{\mathcal{S}}{\rightarrow} M^\infty\) (under \(\mathbb{P}\), as well as under any locally equivalent measure).

(ii) We have shown that \(M_t^T = S_t B_t^\infty \overset{\mathcal{S}}{\rightarrow} M_t^\infty = S_t B_t^\infty\). By Lemma A.2, \(B_t^T \overset{\mathcal{S}}{\rightarrow} B_t^\infty\), and \( B_t^\infty \) is the long bond according to Definition 4.1 (the semimartingale convergence is stronger than the ucp convergence).

Part (iii) is a direct consequence of (i) and (ii).

(iv) Define a new probability measure \(Q^\infty\) by \(Q^\infty|\mathcal{F}_t = M_t^\infty \mathbb{P}|\mathcal{F}_t\) for each \( t \geq 0 \). The distance in total variation between the measures \(Q^T\) for some \( T > 0 \) and \(Q^\infty\) on \(\mathcal{F}_t\) is:
\[
2 \sup_{A \in \mathcal{F}_t} |Q^T(A) - Q^\infty(A)|.
\]
For each $t \geq 0$ we can write:

$$0 = \lim_{T \to \infty} E^Q[|M_T^t - M_t^\infty|] = \lim_{T \to \infty} E^{Q^\infty}\left[ \frac{B_T^T}{B_t^T} - 1 \right].$$

Thus,

$$\lim_{T \to \infty} \sup_{A \in \mathcal{F}_t} \left| E^{Q^\infty}\left[ \frac{B_T^T}{B_t^T} 1_A \right] - E^{Q^\infty}[1_A] \right| = 0.$$

Since

$$\frac{dQ^T}{dQ^\infty}_{\mathcal{F}_t} = \frac{B_T^T}{B_t^T},$$

it follows that

$$\lim_{T \to \infty} \sup_{A \in \mathcal{F}_t} \left| E^{Q^T}[1_A] - E^{Q^\infty}[1_A] \right| = 0.$$

Thus $Q^T$ converge to $Q^\infty$ in total variation on $\mathcal{F}_t$ for each $t$. Since convergence in total variation implies strong convergence of measures, this shows that $Q^\infty$ is the long forward measure according to our Definition 4.2, $Q^\infty = L$. □

### B Proofs of Pricing Kernel Bounds

**Proof of Theorem 5.1.** Let $V$ denote the wealth process of a self-financing portfolio such that $S_t V_t$ is a martingale. By Cauchy-Schwarz inequality we can write:

$$\sigma(S_t) \sigma(V_t) \geq \left| E\left[ \left( \frac{S_T}{S_t} - E\left[ \frac{S_T}{S_t} \right] \right) \left( \frac{V_T}{V_t} - E\left[ \frac{V_T}{V_t} \right] \right) \right] \right|. \quad (B.1)$$

Since $E\left[ \frac{S_T V_T}{S_t V_t} \right] = 1$ by the martingale property, we have

$$E\left[ \left( \frac{S_T}{S_t} - E\left[ \frac{S_T}{S_t} \right] \right) \left( \frac{V_T}{V_t} - E\left[ \frac{V_T}{V_t} \right] \right) \right] = 1 - E[P_T^T]E\left[ \frac{V_T}{V_t} \right].$$

(i) then immediately follows by substituting this result into (B.1), dividing both sides by $\sigma\left( \frac{V_T}{V_t} \right) E\left[ \frac{S_T}{S_t} \right]$ and recognizing that $E\left[ \frac{S_T}{S_t} \right] = E[P_T^T]$. The conditional versions (ii) can be derived in the same way. □

**Proof of Theorem 5.2.** The proof is similar to the proof of Hansen-Jagannathan bounds. By Cauchy-Schwarz inequality,

$$\sigma\left( \frac{M_T^\infty}{M_t^\infty} \right) \sigma\left( \frac{V_T/V_t}{B_T^T/B_t^T} \right) \geq \left| E\left[ \left( \frac{M_T^\infty}{M_t^\infty} - E\left[ \frac{M_T^\infty}{M_t^\infty} \right] \right) \left( \frac{V_T/V_t}{B_T^T/B_t^T} - E\left[ \frac{V_T/V_t}{B_T^T/B_t^T} \right] \right) \right] \right|. $$

Since $E\left[ \frac{S_T V_T}{S_t V_t} \right] = E\left[ \frac{M_T^\infty V_T B_T^T}{M_t^\infty V_t B_t^T} \right] = 1$ and $E[M_T^\infty/M_t^\infty] = E[E_t[M_T^\infty/M_t^\infty]] = 1$, we have

$$E\left[ \left( \frac{M_T^\infty}{M_t^\infty} - E\left[ \frac{M_T^\infty}{M_t^\infty} \right] \right) \left( \frac{V_T/V_t}{B_T^T/B_t^T} - E\left[ \frac{V_T/V_t}{B_T^T/B_t^T} \right] \right) \right] = 1 - E\left[ \frac{M_T^\infty}{M_t^\infty} \right] E\left[ \frac{V_T/V_t}{B_T^T/B_t^T} \right] = 1 - E\left[ \frac{V_T/V_t}{B_T^T/B_t^T} \right],$$

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and (i) immediately follows. 

(ii)-(vi) can be proved in the same way. Note that by Cauchy-Shwarz inequality $E\left[\frac{B_t^\infty}{P_t^\infty}\right] E\left[\frac{B_t^\infty}{P_t^\infty}\right] \geq 1$. This allows us to drop the absolute value in (ii) and similarly in (v).

$\square$

**Proof of Theorem 5.3.** By the martingale property of $S_tV_t$ for the wealth process $V_t$ of any self-financing portfolio and Jensen’s inequality we have:

$$0 = \log E_t[\frac{S_tV_T}{S_tV_t}] \geq E_t[\log \frac{S_tV_T}{S_tV_t}] = E_t[\log \frac{M_t^\infty}{M_t^\infty}] + E_t[\log \frac{V_TB_t^\infty}{V_tB_T^\infty}]. \quad (B.2)$$

Since $M_t^\infty$ is a martingale, $L_t(M_t^\infty/M_t^\infty) = -E_t[\log(M_t^\infty/M_t^\infty)]$ and from (B.2) we immediately obtain (ii). (i) is the unconditional version of (ii).

To prove (iii), we express $L\left(\frac{M_t^\infty}{M_t^\infty}\right)$ in terms of $L\left(\frac{M_t^\infty}{M_t^\infty}\right)$ and then use (i). By definition,

$$L_t\left(\frac{S_t}{S_t}\right) = \log E_t\left[\frac{S_t}{S_t}\right] - E_t[\log \frac{M_t^\infty B_t^\infty}{B_t^\infty M_t^\infty}]$$

$$= \log P_t^T - E_t[\log \frac{B_t^\infty}{B_t^\infty}] + L_t\left(\frac{M_t^\infty}{M_t^\infty}\right)$$

$$= E_t[\log \frac{P_t^TB_t^\infty}{B_t^\infty}] + L_t\left(\frac{M_t^\infty}{M_t^\infty}\right).$$

Taking the expectation on both sides yields

$$EL_t\left(\frac{S_t}{S_t}\right) = E[\log \frac{P_t^TB_t^\infty}{B_t^\infty}] + EL_t\left(\frac{M_t^\infty}{M_t^\infty}\right).$$

Combining it with the fact that $L(X) = EL_t(X) + L(\mathbb{E}_t[X])$ (twice), we have

$$L\left(\frac{S_t}{S_t}\right) = EL_t\left(\frac{S_t}{S_t}\right) + L(\mathbb{E}_t\left[\frac{S_t}{S_t}\right])$$

$$= E[\log \frac{P_t^TB_t^\infty}{B_t^\infty}] + EL_t\left(\frac{M_t^\infty}{M_t^\infty}\right) + L(P_t^T)$$

$$= E[\log \frac{P_t^TB_t^\infty}{B_t^\infty}] + L\left(\frac{M_t^\infty}{M_t^\infty}\right) - L(\mathbb{E}_t\left[\frac{M_t^\infty}{M_t^\infty}\right]) + L(P_t^T)$$

$$= E[\log \frac{P_t^TB_t^\infty}{B_t^\infty}] + L\left(\frac{M_t^\infty}{M_t^\infty}\right) + L(P_t^T).$$

Thus,

$$L\left(\frac{M_t^\infty}{M_t^\infty}\right) = \frac{L\left(\frac{M_t^\infty}{M_t^\infty}\right)}{L\left(\frac{M_t^\infty}{M_t^\infty}\right)} + L(P_t^T) + E[\log \frac{P_t^TB_t^\infty}{B_t^\infty}].$$

Using (i) and the assumption that $E[\log \frac{P_t^TV_t^\infty}{V_t^\infty}] + L(P_t^T) > 0$, if $L(P_t^T) + E[\log \frac{P_t^TB_t^\infty}{B_t^\infty}] \geq 0$,

$$1 \geq \frac{L\left(\frac{M_t^\infty}{M_t^\infty}\right)}{L\left(\frac{M_t^\infty}{M_t^\infty}\right)} \geq \frac{E\left[\log \frac{V_tB_t^\infty}{V_tB_t^\infty}\right]}{E\left[\log \frac{P_t^TV_t^\infty}{V_t^\infty}\right]} + L(P_t^T).$$

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If \( L(P^T_t) + \mathbb{E}[\log \frac{P^T_t B^{\infty}_t}{R^t}] < 0 \),

\[
1 \leq \frac{L(M^{\infty}_t)}{L(S^{\infty}_t)} \leq \frac{\mathbb{E}[\log \frac{V^T_t B^{\infty}_t}{V^t}]}{\mathbb{E}[\log \frac{P^T_t V^t}{V^t}] + L(P^T_t)}.
\]

Combining these two cases yields (iii). (iv) can be shown similarly. □

## C Proofs for Section 6

**Proof of Theorem 6.1.** (i) We first consider the risk-neutral case with \( \gamma = 0 \):

\[
df_t = (Af_t + \alpha_{HJM}(t, \omega, f_t))dt + \sum_{j \in \mathbb{N}} \sigma^j(t, \omega, f_t) dW_Q^{Q,j}.
\] (C.1)

By Assumption 6.1, \( \alpha_{HJM}(t, \omega, h) \) is Lipschitz continuous in \( h \) and uniformly bounded (cf. Filipovic (2001) Lemma 5.2.2). Thus by Theorem 2.4.1 of Filipovic (2001), Eq.(C.1) has a unique continuous weak solution.

Uniqueness of a weak solution with non-zero \( \gamma \) follows by the application of Girsanov’s theorem. We already have uniqueness of a weak solution with \( \gamma = 0 \). By (6.4), \( \gamma \) satisfies the Novikov’s condition (cf. Filipovic (2001) Lemma 2.3.2). Thus, we can define a new measure \( \mathbb{P} \) by

\[
\mathbb{P} |_{\mathcal{F}_t} = \exp \left( - \frac{1}{2} \int_0^t \| \gamma_s \|^2_2 ds - \int_0^t \gamma_s \cdot dW_Q^Q \right) Q |_{\mathcal{F}_t}.
\]

Then by Girsanov’s theorem for infinite-dimensional Brownian motion (cf. Filipovic (2001))

\[
W^P_t = W^Q_t + \int_0^t \gamma_s ds
\] (C.2)

is an infinite-dimensional standard Brownian motions under \( \mathbb{P} \). Thus, \( f_t \) is a unique weak solution of the HJM equation (6.2) under \( \mathbb{P} \) with general \( \gamma \).

(ii) By Filipovic (2001) Theorem 5.2.1, zero-coupon bond price processes \( (P^T_t / A_t)_{t \geq 0} \) taken relative to the process \( A_t = e^{\int_0^t f_s(0) ds} \) are \( Q \)-martingales. This immediately yields the risk-neutral factorization of the pricing kernel under \( \mathbb{P} \).

(iii) Filipovic (2001) Eq.(4.17) gives

\[
\frac{P^T_t}{P^0_t} = A_t \exp \left( - \int_0^t \sigma_s^T \cdot dW^Q_s - \frac{1}{2} \int_0^t \| \sigma_s^2 \|^2_2 ds \right).
\]

Using Eq.(C.2) gives the bond dynamics under \( \mathbb{P} \). □

**Proof of Theorem 6.2.** The proof consists of two parts. We first prove that the process on the right hand side of (6.8) is well defined (the integrals in the exponential are well defined). Next we prove that Eq.(4.1) in Theorem 4.1 holds with \( M^\infty_t = S_t B^{\infty}_t \) and \( B^{\infty}_t \) defined by the right hand side of (6.8). This ensures that the process defined by the right hand side of (6.8) is the long bond \( B^{\infty}_t \) and \( \mathbb{L} \) defined by \( \frac{d\mathbb{L}}{d\mathbb{P}} |_{\mathcal{F}_t} = M^\infty_t \) is the long forward measure. The expression for \( W^L_t \) then follows from Girsanov’s Theorem (cf. Filipovic (2001) Theorem 2.3.3), and the SDE for \( f_t \) under \( \mathbb{L} \) then follows immediately.

We first prove the following lemma which is central to all of the subsequent estimates.
Lemma C.1. The following estimate holds for any function $h \in H^0_w$:

$$
\int_T^\infty |h(x)| dx \leq C(T)\|h\|_{w'}, \quad \text{where } C(T) = K(T^{-\epsilon/2} \wedge 1)
$$

for some $K > 0$ and $\epsilon > 0$.

Proof. Since $w'(x) \geq 1$, for all $h \in H^0_w$ we can write

$$
|h(x)| = |\int_x^\infty h'(s)ds| \leq \|h\|_{w'}(\int_x^\infty \frac{ds}{w'(s)})^{1/2} \leq \|h\|_{w'}(\int_x^\infty K(s^{-(3+\epsilon)}\wedge 1)ds)^{1/2} \leq \|h\|_{w'}K(x^{-(1+\epsilon/2)}\wedge 1),
$$

where the constant $K$ can change from step to step. Thus $\int_T^\infty |h(x)| dx < \|h\|_{w'}K(T^{-\epsilon/2} \wedge 1)$. 

The next lemma ensures that the RHS of (6.8) is well defined.

Lemma C.2. $\int_0^t \|\sigma^\infty_s\|_{L^2_v}^2 ds \leq C^2(0)tD^2_2$.

Proof. By Lemma C.1, $\int_0^\infty |\sigma^\infty_s(u)| du \leq C(0)\|\sigma^\infty_s\|_{H_w}$. This implies

$$
\int_0^t \|\sigma^\infty_s\|_{L^2_v}^2 ds \leq \int_0^t \sum_{j \in \mathbb{N}} \left[ \int_0^\infty |\sigma^\infty_s(u)| du \right]^2 ds \\
\leq \int_0^t \sum_{j \in \mathbb{N}} C^2(0)\|\sigma^\infty_s\|_{w'}^2 ds \\
= C^2(0) \int_0^t \|\sigma^\infty_s\|_{L^2_v(H_w)}^2 ds \\
\leq C^2(0)tD^2_2.
$$

By above lemma, the last integral in (6.8) is well defined. The stochastic integral $\int_0^t \sigma^\infty_s \cdot dW^P_s$ is well defined due to Itô’s isometry. The first integral is bounded by

$$
\frac{1}{2} \int_0^t (\|\gamma_s\|_{L^2}^2 + \|\sigma^\infty_s\|_{L^2_v}^2) ds \leq \frac{1}{2} \int_0^t \Gamma(s)^2 ds + \frac{1}{2} C^2(0)tD^2_2,
$$

which is well defined by the fact that $\Gamma \in L_2(\mathbb{R}_+)$. Thus the right hand side of (6.8) is well defined.

We now re-write $\frac{P_T}{P_0}$ and $B_t^\infty$ in terms of $Q$-Brownian motion $W_t^Q$:

$$
\frac{P_T}{P_0} = A_t \exp \left( - \int_0^t \sigma^T_s \cdot dW^Q_s - \frac{1}{2} \int_0^t \|\sigma^T_s\|_{L^2_v}^2 ds \right),
$$

$$
B_t^\infty = A_t \exp \left( - \int_0^t \sigma^\infty_s \cdot dW^Q_s - \frac{1}{2} \int_0^t \|\sigma^\infty_s\|_{L^2_v}^2 ds \right).
$$

Fix the current $t \geq 0$. By Remark 4.1, showing Eq.(4.1) is equivalent to showing that

$$
\lim_{T \to \infty} \mathbb{E}^Q \left[ \frac{P_T}{P_0} \frac{B_T^\infty}{A_t} - \frac{B_t^\infty}{A_t} \right] = 0. \quad (C.3)
$$
We first introduce some notation. For $v \in [0, t]$ and $T \in [t, \infty]$ define
\[
\bar{J}_v^T := \int_0^v \sigma_s^T \cdot dW_s^Q, \quad \bar{k}_v^T := \frac{1}{2} \int_0^v \|\sigma_s^T\|^2_{2} ds, \\
\bar{\sigma}_v^T := \int_{T-v}^\infty \sigma_v(u) du, \quad \bar{z}_v^T := \frac{1}{2} \int_0^v \|\bar{\sigma}_s^T\|^2_{\mathbb{Q}} ds, \quad Y_v^T := e^{-(\bar{J}_v^T - j_v^\infty) - \bar{z}_v^T}.
\]

For $p \geq 1$ and a random variable $X$ we denote
\[
\|X\|_p := \left(\mathbb{E}^Q[|X|^p]\right)^{1/p},
\]
as long as the expectation is well defined. Then Eq.(C.3) can be re-written as
\[
\lim_{T \to \infty} \|e^{-\bar{J}_v^T - \bar{k}_v^T} - e^{-j_v^\infty - k_v^\infty}\|_1 = 0.
\]
By Hölder’s inequality,
\[
\lim_{T \to \infty} \|e^{-\bar{J}_v^T - \bar{k}_v^T} - e^{-j_v^\infty - k_v^\infty}\|_1 \leq \lim_{T \to \infty} \|e^{-j_v^\infty - k_v^\infty}\|_2 \|e^{-(\bar{J}_v^T - j_v^\infty) - (\bar{k}_v^T - k_v^\infty)} - 1\|_2.
\]
Lemma C.3 and C.4 below show that $\|e^{-j_v^\infty - k_v^\infty}\|_2$ is finite and $\lim_{T \to \infty} \|e^{-(\bar{J}_v^T - j_v^\infty) - (\bar{k}_v^T - k_v^\infty)} - 1\|_2 = 0$, respectively.

**Lemma C.3.** For each $t > 0$, there exists $C$ such that
\[
\sup_{v \leq t} \|Y_v^T\|_2 \leq C < \infty \text{ and } \|e^{-j_v^\infty - k_v^\infty}\|_2 \leq C < \infty.
\]

**Proof.** We begin by considering the process $(Y_v^T)^2 = e^{-(2\bar{J}_v^T - 2j_v^\infty) - 4z_v^T + 2z_v^T}$ for $t \in [0, T]$. By Itô’s formula, $e^{-(2\bar{J}_v^T - 2j_v^\infty) - 4z_v^T}$ is a local martingale. Since it is also positive, it is a supermartingale (in fact, it is a true martingale due to Lemma C.2 and Novikov’s criterion). Therefore for all $v \leq t$,
\[
\mathbb{E}^Q[e^{-(2\bar{J}_v^T - 2j_v^\infty) - 4z_v^T}] \leq 1.
\]
By Lemma C.2, $|z_v^T| \leq |k_v^\infty| \leq \frac{1}{2} C^2(0) t D_2^2$. Thus $\|Y_v^T\|_2^2 = \mathbb{E}^Q[e^{-(2\bar{J}_v^T - 2j_v^\infty) - 4z_v^T + 2z_v^T}] \leq e^{C^2(0) t D_2^2}$. This implies
\[
\sup_{v \leq t} \|Y_v^T\|_2 \leq e^{\frac{1}{2} C^2(0) t D_2^2}.
\]
Similarly, $(e^{-j_v^\infty - k_v^\infty})^2 = e^{-2j_v^\infty - 4k_v^\infty + 2k_v^\infty}$. The process $e^{-2j_v^\infty - 4k_v^\infty}$ is a supermartingale, and $k_v^\infty \leq \frac{1}{2} C^2(0) t D_2^2$ (Lemma C.2). Thus, $\|e^{-j_v^\infty - k_v^\infty}\|_2 \leq e^{\frac{1}{2} C^2(0) t D_2^2}$. \qed

**Lemma C.4.**
\[
\lim_{T \to \infty} \|e^{-(\bar{J}_v^T - j_v^\infty) - (\bar{k}_v^T - k_v^\infty)} - 1\|_2 = 0. \tag{C.4}
\]

**Proof.** We need the following two intermediate lemmas.

**Lemma C.5.** For $T \geq t$, $\sup_{v \leq t} |k_v^T - k_v^\infty| \leq C(0)(C(T - t)t D_2^2)$.
Proof.

\[
\sup_{v \leq t} |k_v^T - k_v^\infty| = \sup_{v \leq t} \left| \frac{1}{2} \sum_{j \in \mathbb{N}} \int_0^v \left( \int_0^{T-s} + \int_0^\infty \right) \sigma_j^2(u) du \left( \int_{T-s}^\infty \sigma_j^2(u) du \right) ds \right|
\]

\[
\leq \sum_{j \in \mathbb{N}} \int_0^t \left( \int_0^\infty |\sigma_j^2(u)| du \right) \left( \int_{T-s}^\infty |\sigma_j^2(u)| du \right) ds
\]

\[
\leq \sum_{j \in \mathbb{N}} \int_0^t C(0)\|\sigma_j^2\|_{\mathcal{W}} C(T-s)\|\sigma_j^2\|_{\mathcal{W}} ds
\]

\[
= C(0)C(T-t) \sum_{j \in \mathbb{N}} \|\sigma_j^2\|_{\mathcal{W}}^2 ds
\]

\[
= C(0)C(T-t) \int_0^t \|\sigma_j^2\|^2_{L^2(H_{\mathcal{W}})} ds
\]

\[
\leq C(0)C(T-t)t\mathcal{D}_2^2.
\]

\[\square\]

Lemma C.6.

\[
\lim_{T \to \infty} \|Y_t^T - 1\|_2 = 0. \quad (C.5)
\]

Proof. By Itô’s formula,

\[
Y_t^T = 1 + \int_0^t Y_v^T \tilde{\sigma}_v^T \cdot dW_v^Q.
\]

By Itô’s isometry, we have

\[
\|Y_t^T - 1\|_2^2 = \mathbb{E}^Q \left( \int_0^t \|Y_v^T \tilde{\sigma}_v^T\|^2_{\mathcal{W}} dv \right).
\]

By Lemma C.1, \(|\tilde{\sigma}_v^{T,j}| \leq C(T-v)\|\sigma_j^2\|_{\mathcal{W}}^2\). Thus

\[
\|Y_t^T - 1\|_2^2 \leq \mathbb{E}^Q \left( \sum_{j \in \mathbb{N}} \int_0^t |Y_v^T|^2 C^2(T-v)\|\sigma_j^2\|^2_{\mathcal{W}} dv \right)
\]

\[
\leq C^2(T-t)\mathbb{E}^Q \left( \int_0^t |Y_v^T|^2 \sum_{j \in \mathbb{N}} \|\sigma_j^2\|^2_{\mathcal{W}} dv \right)
\]

\[
= C^2(T-t)\mathbb{E}^Q \left( \int_0^t |Y_v^T|^2 \|\sigma_j^2\|^2_{L^2(H_{\mathcal{W}})} dv \right)
\]

\[
\leq C^2(T-t)\mathbb{E}^Q \left( \int_0^t |Y_v^T|^2 \mathcal{D}_2^2 dv \right)
\]

\[
\leq C^2(T-t)\mathbb{E}^Q (|Y_v^T|^2) dv
\]

\[
\leq C^2(T-t)\mathbb{E}^Q (|Y_v^T|^2) dv \quad (\text{Lemma C.3})
\]

\[
= C^2(T-t)\mathcal{D}_2^2 C^2 t.
\]

Since \(\lim_{T \to \infty} C(T-t) = 0\), Eq.(C.5) is verified. \[\square\]
Now we return to the proof of Lemma C.4.

\[ \| e^{-(j^T-j^\infty)-(k^T-k^\infty)} - 1 \|_2 = \| Y^T_t e^{-j^T-\lambda t} - 1 \|_2 \leq \| (Y^T_t - 1) e^{-j^T-\lambda t} \|_2 + \| e^{-j^T-\lambda t} \|_2. \]

Recall that by Lemma C.5, \( |k^T_t - k^\infty| \leq C(0)C(T-t)tD_2^2 \). Using the same approach as Lemma C.2, we can show that \( |z^T_i| \leq \frac{1}{2} C^2(T-t)tD_2^2 \). Thus, we have

\[ \| e^{-(j^T-j^\infty)-(k^T-k^\infty)} - 1 \|_2 \leq \| Y^T_t - 1 \|_2 e^{\frac{1}{2} C^2(T-t)tD_2^2 + C(0)C(T-t)tD_2^2} + e^{\frac{1}{2} C^2(T-t)tD_2^2 + C(0)C(T-t)tD_2^2} - 1 \]

Finally Eq.(C.4) is verified using Lemma C.6 and the fact that \( \lim_{T \to \infty} C(T-t) = 0 \).

**D Proof of Theorem 8.1**

Let \( Q^\pi(t, x, \cdot) \) denote the transition measure of \( X \) under \( Q^\pi \). We first prove that Assumption 8.1 implies \( \int_E \zeta(dy) \frac{1}{\pi_R(y)} < \infty \). Since we assume \( \mathbb{E}_x[S_t] < \infty, P^0_t < \infty \) for all \( t > 0 \). Rewrite the zero-coupon bond price under \( Q^\pi \):

\[ P^0_t = e^{-\lambda R t} \pi_R(x) \mathbb{E}_x^{Q^\pi} \left[ \frac{1}{\pi_R(X_t)} \right] < \infty. \]

Thus,

\[ \mathbb{E}_x^{Q^\pi} \left[ \frac{1}{\pi_R(X_t)} \right] = \int_E Q^\pi(t, x, dy) \frac{1}{\pi_R(y)} < \infty \]

for all \( t > 0 \) and \( x \in E \). This implies

\[ \int_E \zeta(dy) \frac{1}{\pi_R(y)} = \int_E Q^\pi(t, x, dy) \frac{1}{\pi_R(y)} + \int_E \left( \zeta(dy) - Q^\pi(t, x, dy) \right) \frac{1}{\pi_R(y)} dy \leq \int_E Q^\pi(t, x, dy) \frac{1}{\pi_R(y)} + \frac{c}{\pi_R(x)} e^{-\alpha t} < \infty. \]

We now verify Eq.(4.1) with \( M^\infty_t = M^R_t \). This then identifies \( e^{\lambda R t} \frac{\pi_R(X_t)}{\pi_R(X_0)} \) with the long bond \( B^\infty \) and \( Q^\pi = \mathbb{L} \). We choose to verify it under \( Q^\pi \), i.e. Eq.(4.3) with \( Q^V = Q^\pi \) due to its convenient form. Since \( P^0_t = e^{-\lambda R (T-t)} \pi_R(X_t) \mathbb{E}_{X_t}^{Q^\pi} \left[ \frac{1}{\pi_R(X_T)} \right] \), we have

\[ e^{-\lambda R t} \frac{P^0_t}{\pi_R(X_0)} \frac{\pi_R(X_0)}{\pi_R(X_t)} = \frac{\mathbb{E}_{X_t}^{Q^\pi} \left[ \frac{1}{\pi_R(X_T)} \right]}{\mathbb{E}_{X_0}^{Q^\pi} \left[ \frac{1}{\pi_R(X_T)} \right]}. \]

Let \( J := \int_E \zeta(dy) \frac{1}{\pi_R(y)} \). Since \( \mathbb{E}_{X_t}^{Q^\pi} \left[ \frac{1}{\pi_R(X_T)} \right] \leq \mathbb{E}_{X_t}^{Q^\pi} \left[ \frac{1}{\pi_R(X_{T-t})} \right] \leq J + \frac{c}{\pi_R(X_t)} e^{-\alpha (T-t)}, \)

(D.1)
and for each initial state \( X_0 = x \in E \):

\[ J - \frac{c}{\pi_R(x)} e^{-\alpha T} \leq \mathbb{E}_{X_t}^{Q^\pi} \left[ \frac{1}{\pi_R(X_T)} \right] \leq J + \frac{c}{\pi_R(x)} e^{-\alpha T}. \]

(D.2)
For each $x \in E$ there exists $T_0$ such that for $T \geq T_0$, $\frac{c}{\pi R(x)} e^{-\alpha T} \leq J/2$. We can thus write for each $x \in E$:

$$-1 \leq e^{-\lambda t} \frac{P_t^T \pi_R(x)}{P_0^T \pi_R(X_t)} - 1 \leq \frac{2}{J} \left( \frac{c}{\pi_R(X_t)} e^{-\alpha(T-t)} + \frac{c}{\pi_R(x)} e^{-\alpha T} \right),$$

Thus,

$$\left| e^{-\lambda t} \frac{P_t^T \pi_R(x)}{P_0^T \pi_R(X_t)} - 1 \right| \leq \frac{2}{J} \left( \frac{c}{\pi_R(X_t)} e^{-\alpha(T-t)} + \frac{c}{\pi_R(x)} e^{-\alpha T} \right) + 1.$$

Since for each $t$ the $\mathcal{F}_t$-measurable random variable $\frac{1}{\pi_R(X_t)}$ is integrable under $\mathbb{Q}^x_0$ for each $x \in E$, for each $t$ the $\mathcal{F}_t$-measurable random variable $\left| e^{-\lambda t} \frac{P_t^T \pi_R(x)}{P_0^T \pi_R(X_t)} - 1 \right|$ is bounded by an integrable random variable. Furthermore, by Eq.(D.1) and (D.2),

$$\lim_{T \to \infty} \left| e^{-\lambda t} \frac{P_T^T(\omega) \pi_R(x)}{P_0^T \pi_R(X_t(\omega))} - 1 \right| = 0$$

for each $\omega$. Thus, by the Dominated Convergence Theorem Eq.(4.3) is verified with $B_t^\infty = e^{\lambda t} \frac{\pi_R(X_t)}{\pi_R(X_0)}$. \(\square\)

### E Verification of Theorem 4.1 for Markovian Examples

#### E.1 Vasicek Model

The bond price in Vasicek model is given by

$$P_t^T = P(X_t, T - t) = A(T - t) e^{-X_t B(T-t)}, \quad \text{(E.1)}$$

where

$$B(\tau) = \frac{1}{\kappa} e^{-\kappa \tau}, \quad A(\tau) = \exp \left\{ (\theta - \frac{\sigma^2}{2\kappa^2})(B(\tau) - \tau) - \frac{\sigma^2}{4\kappa} B^2(\tau) \right\}. $$

We can verify directly that

$$\lim_{T \to \infty} \frac{P(y, T - t)}{P(x, T)} = e^{(\theta - \frac{\sigma^2}{2\kappa^2})t - \frac{1}{\kappa} (y-x)}. $$

It remains to verify Eq.(4.3) under $\mathbb{Q}^x_0$ with $B_t^\infty = e^{(\theta - \frac{\sigma^2}{2\kappa^2})t - \frac{1}{\kappa} (X_t - X_0)}$. By Eq.(E.1),

$$\frac{P_t^T}{P_0^T B_t^\infty} = \exp \left( \left( \theta - \frac{\sigma^2}{2\kappa^2} \right)(\epsilon^T - e^{-\kappa(T-t)}) - \frac{\sigma^2}{4\kappa} (e^{-2\kappa(T-t)} - e^{-2\kappa T} - e^{-2\kappa(T-t)} + 2e^{-\kappa T}) + X_t \frac{e^{-\kappa(T-t)}}{\kappa} - X_0 \frac{e^{-\kappa T}}{\kappa} \right).$$

It is easy to verify that

$$\lim_{T \to \infty} \left| \frac{P_T^T(\omega)}{P_0^T B_T^\infty(\omega)} - 1 \right| = 0 \quad \text{(E.2)}$$

for each $\omega$. On the other hand, for any constant $\epsilon > 0$ there exists $T_0 > 0$ such that for $T \geq T_0$

$$\left| \frac{P_t^T}{P_0^T B_t^\infty} - 1 \right| < (1 + \epsilon) e^{\epsilon X_t} + 1.$$
E.2 Square-root Model with Absorption at Zero

Consider a CIR process (8.5) under \( Q \) with \( a = 0 \). The CIR bond price is given by

\[
P_T^t = e^{-X_t B(T-t)} \quad \text{with} \quad B(t) = \frac{2(\kappa-1)}{\gamma + (\kappa+\gamma)(\kappa+\gamma)} \quad \text{and} \quad \gamma = \sqrt{\kappa^2 + 2\sigma^2}.
\]

We can verify directly that

\[
\lim_{T \to \infty} \frac{P(y, T-t)}{P(x, T)} = e^{\frac{\kappa}{\sigma^2}(y-x)}.
\]

By the affine property of the CIR model, we can also show that \( \pi(x) = e^{\frac{\kappa}{\sigma^2}x} \) is an eigenfunction of the CIR pricing semigroup with \( a = 0 \) with the eigenvalue \( \lambda = 0 \). We now verify Eq.(4.3) under \( Q^V = Q^x \) with \( B_t^\infty = e^{\frac{\kappa}{\sigma^2}(X_t-X_0)} \). First observe that

\[
\frac{P_T^t}{P_0^t B_t^\infty} = e^{\left(\frac{\kappa}{\sigma^2} - B(T-t)\right)X_t - \left(\frac{\kappa}{\sigma^2} - B(T)\right)X_0}.
\]

Since

\[
\lim_{\tau \to \infty} \left(\frac{\gamma - \kappa}{\sigma^2} - B(\tau)\right) = 0,
\]

we have

\[
\lim_{T \to \infty} \left| \frac{P_T^t(\omega)}{P_0^t V_t(\omega)} - 1 \right| = 0 \tag{E.3}
\]

for each \( \omega \). On the other hand, for any constant \( \epsilon > 0 \) there exists \( T_0 > 0 \) such that for \( T \geq T_0 \)

\[
\left| \frac{P_T^t}{P_0^t V_t} - 1 \right| \leq (1 + \epsilon)e^{\epsilon X_t} + 1.
\]

By the affine property of the CIR model, there exists \( \epsilon > 0 \) such that \( e^{\epsilon X_t} \) is integrable. Thus, by Eq.(E.3) and the Dominated Convergence Theorem Eq.(4.3) holds with \( B_t^\infty = e^{\frac{\kappa}{\sigma^2}(X_t-X_0)} \) and \( L = Q^x \), even though \( \pi = e^{\frac{\kappa}{\sigma^2}x} \) is not a recurrent eigenfunction.

References


