1. Forward and Futures

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We consider only one “underlying” risky security (it can be a stock or exchange rate), and we use $S$ to
denote its price, with $S_0$ being its current price (known) and $S_T$ being its price at time $T$ (unknown). Also,
ignore bid-ask spread and transaction cost and assume that you can buy or sell any amount of the security at
the price $S_0$. To be concrete, let’s set $S_0 = 100$.

1. Suppose you are long one share (you own one share) of this underlying security, plot your portfolio’s
payoff at time $T$. Assume that the security does not pay dividend or earn interest.

Answer: The payoff is simply $S_T$. You get what the security is worth at time $T$.

![Graph 1]

2. Suppose you are long one share (you own one share) of this underlying security, plot your portfolio’s
payoff at time $T$. Assume that the security pays $5 dividend at $T$ (and nothing in between).

Answer: In this case, you not only can sell the stock for its market price at time $T$, which is $S_T$, but you
also receive a $5 dividend. Hence, the payoff at time $T$ is $S_T + 5$.

![Graph 2]

Note in the graph, when $S_T = 100$, the payoff is 105.

3. Suppose you are long ten shares of this underlying security (and nothing else), plot your portfolio’s
payoff at time $T$.

Answer: The payoff is ten times what you get previously: $10(S_T + 5)$.
How much shall you sell this portfolio for? Assume that the security earns continuously compounded rate of 5% per year and \( T = 2 \) years.

**Answer:** The question is a bit vague. If the question is how much you can sell this portfolio for right now, it is simply the current market price, which is $100 per share, and hence $1000 for the ten shares.

If I reformulate the question to be: “What is the cost of buying this portfolio and carrying it over to \( T = 2 \) years?” Then, the answer is \( 10 \times (100e^{0.05\times2} - 5) = 1055.2 \). The initial cost of buying the ten stocks is simply $1000, but if you borrow this amount and pay the money back 2 years later at a 5% rate, that cost grows to $1105.2 at \( T = 2 \) years. However, if you hold this stock for two years, you also end up receiving a $50 dividend. That reduces your cost by $50.

4. Plot the time-\( T \) payoff of a zero-coupon bond with maturity date \( T \) and par value of $1 as a function of this underlying security price at time \( T \).

**Answer:** The payoff is $1, regardless of what the underlying price is.

Also, if the continuously compounding interest rate is 5% per year, how much is this bond worth today is \( T = 2 \)?

**Answer:** That’s just the present value of $1, which is \( 1e^{-0.05\times2} = 0.9048 \).

5. Suppose you borrow \( S_0 = 100 \) dollars to buy one share of this security today, the lending rate on the dollar is 5% per year (continuous compounding). The security does not pay dividend or earn interest. Plot the time-\( T \) payoff of your portfolio. Also explain what your portfolio is composed of.
**Answer:** The payoff includes (i) paying back the borrowed money plus interest, which is $100e^{0.05*T}$, and (ii) selling the stock for $S_T$. Hence, the payoff is $S_T - 100e^{0.05*T}$.

For concreteness, let me just set $T = 2$ and get the payoff to be $S_T - 100e^{0.05*2} = S_T - 110.52$.

If you are REALLY worried about the grade, you get full credit if you just illustratively show $S_T - 100e^{0.05*2}$, or set $T$ to any number just for concreteness.

(a) If there are forward contracts traded on this security with the expiry $T = 2$, what should be the fair forward price?

**Answer:** The fair forward price should be equal to the cost of buying and carrying forward 1 share to expiry at time $T$. That cost is imply $100e^{0.05*2} = 110.52$ because there is just the initial buying cost plus interest.

(b) Design a trading strategy to make money if the forward price is $105 or $115, respectively.

**Answer:** If the forward price is $105, which is lower than the buy-and-carry cost, you should enter a long position in the forward contract at the quoted forward price of $105, so that you can buy the security at $105 at time $T$. Then, you should also enter a short position in the “buy-and-carry strategy.” Executing the long forward strategy is simple (just sign the contract and shake hands, and you don’t need to pay or receive anything because the contract is signed at the forward price and hence is worth zero). Executing the shorting part needs a little more detail: You first borrow this security (from the broker or your friend), and sell this security to the market for $100. Then, you save this money in the bank for 2 years to make interest. The money will grow to $100e^{0.05*2} = 110.52$ at $T = 2$. You use $105$ of this saved money to buy the security according to your forward contract, and return the security back to your broker or friend. You have $110.52 - 105 = $5.52 left. That’s your net money, with all positions settled.

If the forward price is $115, which is higher than the buy-and-carry cost, you should do the reverse of the above: You should enter a short position in the forward contract at the quoted forward price of $115, so that you can sell the security at $115 at time $T$. Again, this is just sign and handshake. Then, you should also enter a long position in the “buy-and-carry strategy,” which is just buying and carrying: Borrow $100, buy the security, carry it to time $T$. At time $T$, you sell the stock you carried over for $115 according to your forward agreement. Then, you pay back your borrowed money, which is $110.52 (100 plus interest). You net $4.48 dollars.

(c) Suppose your portfolio is just long a forward contract expiring at $T = 2$ and with a delivery price ($K$) of $100$, what is the value of your portfolio?
Answer: The forward price is $F = 110.52$, which means that the forward contract would be worth zero today if the delivery price ($K$) is set to this forward price. However, the actual delivery price $K = 100$, which differs from the forward price. Thus, this contract has value. The difference between the forward price and the delivery price is $F - K = 110.52 - 100 = 10.52$. This is the worth of the forward contract at expiry $T$. Its current value is $e^{-0.05\times2}(F - K) = 9.52$.

What’s the value if the delivery price is $120$?
Answer: The value of the contract is $e^{-0.05\times2}(F - K) = e^{-0.05\times2}(110.52 - 120) = -8.58$, which means that if you want to enter a long position in this contract, you will receive $8.58–That is, the cost for the long position is negative (negative cost means positive cash flow).

6. Repeat the question, but assume that the security also earns a continuously compounding interest rate of 2%.

(a) If there are forward contracts traded on this security with the expiry $T = 2$, what should be the fair forward price?
Answer: The fair forward price should be equal to the cost of buying and carrying forward 1 share to expiry at time $T$. The cost of buying one share is $100e^{0.05\times2} = 110.52$, which is just the initial buying cost plus interest. However, since this security grows at 2% rate, you don’t need to buy one share, you just need to buy $e^{-0.02\times2} = 0.96$ share and it will grow at 2% rate into one share at time $T$. The cost of this 0.96 share is: $0.96 \times 110.52$, or in full equation, $e^{-0.02\times2}100e^{0.05\times2} = 100e^{2\times0.03} = 106.18$.

Under continuous compounding, you can directly take the difference between the “cost” rate and the “benefit” rate and just compute the net cost rate, which is 3% in this case.

(b) Design a trading strategy to make money if the forward price is $105 or $115, respectively.
Answer: If the forward price is $105, which is lower than the buy-and-carry cost of $106.18, you should enter a long position in the forward contract at the quoted forward price of $105, so that you can buy the security at $105 at time $T$. Then, you should also enter a short position in the “buy-and-carry strategy.” Executing the long position is simple (just sign the contract and shake hands, and you don’t need to pay or receive anything because the contract is signed at the forward price and hence is worth zero). Executing the shorting part needs a little more detail: You first borrow 0.9608 share of this security (from the broker or your friend), and sell this security to the market for $100 per share, which is $96.08 for the 0.9608 share. Then, you save this money in the bank for 2 years to make interest. That’ll be $100e^{(0.05 - 0.02)\times2} = 106.18$, as we calculated before.

At time $T$, you use $105$ of this saved money to buy the security according to your forward contract, and return 1 full share of the security back to your broker or friend— Although you borrowed 0.96 share, but the share grew (say due to stock dividend) and you need to return both the original share and the grown share to the lender, which comes up as one share. Your bank account will have $106.18 - 105 = 1.18$ left.

If the forward price is $115, which is higher than the buy-and-carry cost, you should do the reverse of the above: You should enter a short position in the forward contract at the quoted forward price of $115, so that you can sell the security at $115 at time $T$. Again, this is just sign and handshake. Then, you should enter a long position in the “buy-and-carry strategy,” which is just buying 0.96 share and letting it to grow to 1 share: Borrow $100e^{-0.02\times2} = 96.08$, buy 0.9608 of the security, carry it to time $T$, and let it become 1 share.
At time $T$, you sell the security, which has grown to 1 share, for $115 according to your forward agreement. Then, you pay back your borrowed money, which is $106.18 ($96.08 plus interest). You net $8.82.

(c) Suppose your portfolio is just long a forward contract expiring at $T = 2$ and with a delivery price ($K$) of $100, what is the value of your portfolio?

**Answer:** The forward price is $F = 106.18$, which means that the forward contract would be worth zero today if the delivery price ($K$) is set to this forward price. The value of the contract is the present value of the difference between the forward price and the delivery price, $e^{-0.05 \times 2} (F - K) = e^{-0.05 \times 2} (106.18 - 100) = 5.59$.

What’s the value if the delivery price is $120?  
**Answer:** $e^{-0.05 \times 2} (F - K) = e^{-0.05 \times 2} (106.18 - 120) = -12.50$

(d) Repeat the question, but assume that the security has quarterly dividend payments of $2 for each share. And assume that you are at the beginning of a quarter so that you’ll have 8 dividend payments over the next two years.

**Answer:** First, you need to figure out the forward price using buy-and-carry argument. All other parts are pretty much repetition.

You want to end up with one share at time $T$. This time, you do need to buy one share now because the share does not grow. The cost of buying this one share is $100e^{0.05 \times 2} = 110.52$ (initial cost of 100 plus interest). In addition, owning this share leads to dividend benefits, which we need to deduct for the cost. There are 8 dividends and you need to compute their value at $T$:

$$2e^{0.05 \times 7/4} + 2e^{0.05 \times 6/4} + 2e^{0.05 \times 5/4} + 2e^{0.05 \times 4/4} + 2e^{0.05 \times 3/4} + 2e^{0.05 \times 2/4} + 2e^{0.05 \times 1/4} + 2 = 16.72$$

**Hence, the forward price is 110.52 – 16.72 = 93.80.** (actually 93.7947 to be exact if we do not round off the intermediate numbers).