Pricing and Hedging under the Black-Merton-Scholes Model

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Options Markets
Outline

1. Primer on continuous-time processes
2. BMS option pricing
3. Hedging under the BMS model
   - Delta
   - Vega
   - Gamma
The Black-Scholes-Merton (BSM) model

- Black and Scholes (1973) and Merton (1973) derive option prices under the following assumption on the stock price dynamics,

\[ dS_t = \mu S_t dt + \sigma S_t dW_t \]

- The binomial model: Discrete states and discrete time (The number of possible stock prices and time steps are both finite).

- The BSM model: Continuous states (stock price can be anything between 0 and \(\infty\)) and continuous time (time goes continuously).

- Scholes and Merton won Nobel price. Black passed away.

- BSM proposed the model for stock option pricing. Later, the model has been extended/twisted to price currency options (Garman&Kohlhagen) and options on futures (Black).

- I treat all these variations as the same concept and call them indiscriminately the BMS model.
Primer on continuous time process

\[ dS_t = \mu S_t dt + \sigma S_t dW_t \]

- The driver of the process is \( W_t \), a Brownian motion, or a Wiener process.
  - The sample paths of a Brownian motion are continuous over time, but nowhere differentiable.
  - It is the idealization of the trajectory of a single particle being constantly bombarded by an infinite number of infinitesimally small random forces.
  - If you sum the absolute values of price changes over a day (or any time horizon) implied by the model, you get an infinite number.
  - If you tried to accurately draw a Brownian motion sample path, your pen would run out of ink before one second had elapsed.

- The first who brought Brownian motion to finance is Bachelier in his 1900 PhD thesis: The theory of speculation.
Properties of a Brownian motion

\[ dS_t = \mu S_t dt + \sigma S_t dW_t \]

- The process \( W_t \) generates a random variable that is normally
distributed with mean 0 and variance \( t, \phi(0, t) \). (Also referred to as Gaussian.)

\[ Everybody \ believes \ in \ the \ normal \ approximation, \ the \ experimenters \ because \ they \ believe \ it \ is \ a \ mathematical \ theorem, \ the \ mathematicians \ because \ they \ believe \ it \ is \ an \ experimental \ fact! \]

- The process is made of independent normal increments \( dW_t \sim \phi(0, dt) \).
  - “\( d \)” is the continuous time limit of the discrete time difference (\( \Delta \)).
  - \( \Delta t \) denotes a finite time step (say, 3 months), \( dt \) denotes an extremely thin slice of time (smaller than 1 milisecond).
  - It is so thin that it is often referred to as instantaneous.
  - Similarly, \( dW_t = W_{t+dt} - W_t \) denotes the instantaneous increment (change) of a Brownian motion.

- By extension, increments over non-overlapping time periods are independent: For \( (t_1 > t_2 > t_3) \), \( (W_{t_3} - W_{t_2}) \sim \phi(0, t_3 - t_2) \) is independent of \( (W_{t_2} - W_{t_1}) \sim \phi(0, t_2 - t_1) \).
Properties of a normally distributed random variable

\[ dS_t = \mu S_t dt + \sigma S_t dW_t \]

- If \( X \sim \phi(0, 1) \), then \( a + bX \sim \phi(a, b^2) \).
- If \( y \sim \phi(m, V) \), then \( a + by \sim \phi(a + bm, b^2 V) \).
- Since \( dW_t \sim \phi(0, dt) \), the instantaneous price change
  \[ dS_t = \mu S_t dt + \sigma S_t dW_t \sim \phi(\mu S_t dt, \sigma^2 S_t^2 dt) \].
- The instantaneous return \( \frac{dS}{S} = \mu dt + \sigma dW_t \sim \phi(\mu dt, \sigma^2 dt) \).

▶ Under the BSM model, \( \mu \) is the annualized mean of the instantaneous return — instantaneous mean return.
▶ \( \sigma^2 \) is the annualized variance of the instantaneous return — instantaneous return variance.
▶ \( \sigma \) is the annualized standard deviation of the instantaneous return — instantaneous return volatility.
Geometric Brownian motion

\[ \frac{dS_t}{S_t} = \mu dt + \sigma dW_t \]

- The stock price is said to follow a geometric Brownian motion.
- \( \mu \) is often referred to as the drift, and \( \sigma \) the diffusion of the process.
- Instantaneously, the stock price change is normally distributed, \( \phi(\mu S_t dt, \sigma^2 S_t^2 dt) \).
- Over longer horizons, the price change is lognormally distributed.
- The log return (continuous compounded return) is normally distributed over all horizons:
  \[ d \ln S_t = (\mu - \frac{1}{2} \sigma^2) dt + \sigma dW_t. \] (By Itô’s lemma)
  - \( d \ln S_t \sim \phi(\mu dt - \frac{1}{2} \sigma^2 dt, \sigma^2 dt) \).
  - \( \ln S_t \sim \phi(\ln S_0 + \mu t - \frac{1}{2} \sigma^2 t, \sigma^2 t) \).
  - \( \ln S_T / S_t \sim \phi((\mu - \frac{1}{2} \sigma^2)(T - t), \sigma^2 (T - t)) \).
- Integral form: \( S_t = S_0 e^{\mu t - \frac{1}{2} \sigma^2 t + \sigma W_t} \), \( \ln S_t = \ln S_0 + \mu t - \frac{1}{2} \sigma^2 t + \sigma W_t \).
Normal versus lognormal distribution

\[ dS_t = \mu S_t dt + \sigma S_t dW_t, \quad \mu = 10\%, \sigma = 20\%, S_0 = 100, t = 1. \]

The earliest application of Brownian motion to finance is Louis Bachelier in his dissertation (1900) “Theory of Speculation.” He specified the stock price as following a Brownian motion with drift:

\[ dS_t = \mu dt + \sigma dW_t \]
Simulate 100 stock price sample paths

\[ dS_t = \mu S_t \, dt + \sigma S_t \, dW_t, \quad \mu = 10\%, \, \sigma = 20\%, \, S_0 = 100, \, t = 1. \]

- Stock with the return process: \( d \ln S_t = (\mu - \frac{1}{2} \sigma^2) \, dt + \sigma dW_t. \)
- Discretize to daily intervals \( dt \approx \Delta t = 1/252. \)
- Draw standard normal random variables \( \varepsilon(100 \times 252) \sim \phi(0, 1). \)
- Convert them into daily log returns: \( R_d = (\mu - \frac{1}{2} \sigma^2) \Delta t + \sigma \sqrt{\Delta t} \varepsilon. \)
- Convert returns into stock price sample paths: \( S_t = S_0 e^{\sum_{d=1}^{252} R_d}. \)
The key idea behind BMS option pricing

- Under the binomial model, if we cut time step $\Delta t$ small enough, the binomial tree converges to the distribution behavior of the geometric Brownian motion.

- Reversely, the Brownian motion dynamics implies that if we slice the time thin enough ($dt$), it behaves like a binomial tree.
  - Under this thin slice of time interval, we can combine the option with the stock to form a riskfree portfolio, like in the binomial case.
  - Recall our hedging argument: Choose $\Delta$ such that $f - \Delta S$ is riskfree.
  - The portfolio is riskless (under this thin slice of time interval) and must earn the riskfree rate.
  - Since we can hedge perfectly, we do not need to worry about risk premium and expected return. Thus, $\mu$ does not matter for this portfolio and hence does not matter for the option valuation.
  - Only $\sigma$ matters, as it controls the width of the binomial branching.
The hedging proof*

- The stock price dynamics: \(dS = \mu Sdt + \sigma SdW_t\).
- Let \(f(S, t)\) denote the value of a derivative on the stock, by Ito’s lemma: 
  \[df_t = (f_t + f_S \mu S + \frac{1}{2} f_{SS} \sigma^2 S^2) dt + f_S \sigma SdW_t.\]
- At time \(t\), form a portfolio that contains 1 derivative contract and \(-\Delta = f_S\) of the stock. The value of the portfolio is \(P = f - f_SS\).
- The instantaneous uncertainty of the portfolio is \(f_S \sigma SdW_t - f_S \sigma SdW_t = 0\). Hence, instantaneously the delta-hedged portfolio is riskfree.
- Thus, the portfolio must earn the riskfree rate: \(dP = rPdt\).
- \(dP = df - f_S dS = (f_t + f_S \mu S + \frac{1}{2} f_{SS} \sigma^2 S^2 - f_S \mu S) dt = r(f - f_SS)dt\)
- This lead to the fundamental partial differential equation (PDE): 
  \(f_t + rSf_S + \frac{1}{2} f_{SS} \sigma^2 S^2 = rf\).
- If the stock pays a continuous dividend yield \(q\), the portfolio P&L needs to be modified as \(dP = df - f_S(dS + qSdt)\), where the stock investment P&L includes both the capital gain \((dS)\) and the dividend yield \((qSdt)\).
- The PDE then becomes: 
  \(f_t + (r - q)Sf_S + \frac{1}{2} f_{SS} \sigma^2 S^2 = rf\).
- No where do we see the drift (expected return) of the price dynamics (\(\mu\)) in the PDE.
The hedging argument leads to the following partial differential equation:

\[
\frac{\partial f}{\partial t} + (r - q)S \frac{\partial f}{\partial S} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 f}{\partial S^2} = rf
\]

- The only free parameter is \( \sigma \) (as in the binomial model).

Solving this PDE, subject to the terminal payoff condition of the derivative (e.g., \( f_T = (S_T - K)^+ \) for a European call option), BMS derive analytical formulas for call and put option value.

- Similar formula had been derived before based on distributional (normal return) argument, but \( \mu \) was still in.
- More importantly, the derivation of the PDE provides a way to hedge the option position.

The PDE is generic for any derivative securities, as long as \( S \) follows geometric Brownian motion.

- Given boundary conditions, derivative values can be solved numerically from the PDE.
How effective is the BMS delta hedge in practice?

- If you sell an option, the BMS model says that you can completely remove the risk of the call by continuously rebalancing your stock holding to neutralize the delta of the option.

- We perform the following experiment using history S&P options data from 1996 - 2011:
  - Sell a call option
  - Record the P&L from three strategies
    - **Hold**: Just hold the option.
    - **Static**: Perform delta hedge at the very beginning, but with no further rebalancing.
    - **Dynamic**: Actively rebalance delta at the end of each day.

- “D” represents a daily discretization of the BMS suggestion.

- Compared to “H” (no hedge), how much do you think the daily discretized BMS suggestion (“D”) can reduce the risk of the call?

  \[
  [A] 0, \quad [B] 5\%, \quad [C] 50\%, \quad [D] 95\%
  \]
Example: Selling a 30-day at-the-money call option

- Unhedged, the PL variance can be over 100% of the option value.
- Un-rebalanced, the delta hedge deteriorates over time as the delta is no longer neutral after a whole.
- Rebalanced daily, 95% of the risk can be removed over the whole sample period.
- The answer is “D.”
Hedging options at different maturities and time periods

- Dynamic delta hedge works equally well on both short and long-term options.
- Hedging performance deteriorates in the presence of large moves.
The BSM formulae

\[ c_t = S_t e^{-q(T-t)} N(d_1) - Ke^{-r(T-t)} N(d_2), \]
\[ p_t = -S_t e^{-q(T-t)} N(-d_1) + Ke^{-r(T-t)} N(-d_2), \]

where

\[ d_1 = \frac{\ln(S_t/K) + (r-q)(T-t) + \frac{1}{2} \sigma^2(T-t)}{\sigma \sqrt{T-t}}, \]
\[ d_2 = \frac{\ln(S_t/K) + (r-q)(T-t) - \frac{1}{2} \sigma^2(T-t)}{\sigma \sqrt{T-t}} = d_1 - \sigma \sqrt{T-t}. \]

Black derived a variant of the formula for futures (which I like better):

\[ c_t = e^{-r(T-t)} [F_t N(d_1) - KN(d_2)], \]
with \( d_{1,2} = \frac{\ln(F_t/K) \pm \frac{1}{2} \sigma^2(T-t)}{\sigma \sqrt{T-t}}. \)

- Recall: \( F_t = S_t e^{(r-q)(T-t)}. \)
- Once I know call value, I can obtain put value via put-call parity:
  \[ c_t - p_t = e^{-r(T-t)} [F_t - K_t]. \]
Cumulative normal distribution

\[ c_t = e^{-r(T-t)} \left[ F_t N(d_1) - KN(d_2) \right], \quad d_{1,2} = \frac{\ln(F_t/K) \pm \frac{1}{2} \sigma^2(T-t)}{\sigma \sqrt{T-t}} \]

- \( N(x) \) denotes the cumulative normal distribution, which measures the probability that a normally distributed variable with a mean of zero and a standard deviation of 1 (\( \phi(0,1) \)) is less than \( x \).
- See tables at the end of the book for its values.
- Most software packages (including excel) has efficient ways to computing this function.
- Properties of the BSM formula:
  - As \( S_t \) becomes very large or \( K \) becomes very small, \( \ln(F_t/K) \uparrow \infty \), \( N(d_1) = N(d_2) = 1 \). \( c_t = e^{-r(T-t)} [F_t - K] \).
  - Similarly, as \( S_t \) becomes very small or \( K \) becomes very large, \( \ln(F_t/K) \uparrow -\infty \), \( N(-d_1) = N(-d_2) = 1 \). \( p_t = e^{-r(T-t)} [-F_t + K] \).
Implied volatility

\[ c_t = e^{-r(T-t)} [F_t N(d_1) - KN(d_2)], \quad d_{1,2} = \frac{\ln(F_t/K) \pm \frac{1}{2}\sigma^2(T-t)}{\sigma \sqrt{T-t}} \]

- Since \( F_t \) (or \( S_t \)) is observable from the underlying stock or futures market, \((K, t, T)\) are specified in the contract. The only unknown (and hence free) parameter is \( \sigma \).

- We can estimate \( \sigma \) from time series return. (standard deviation calculation).

- Alternatively, we can choose \( \sigma \) to match the observed option price — implied volatility (IV).

- There is a one-to-one correspondence between prices and implied volatilities.

- Traders and brokers often quote implied volatilities rather than dollar prices.

- The BSM model says that \( IV = \sigma \). In reality, the implied volatility calculated from different options (across strikes, maturities, dates) are usually different.
Risk-neutral valuation

- Recall: Under the binomial model, we derive a set of risk-neutral probabilities such that we can calculate the expected payoff from the option and discount them using the risk-free rate.
  - Risk premiums are hidden in the risk-neutral probabilities.
  - If in the real world, people are indeed risk-neutral, the risk-neutral probabilities are the same as the real-world probabilities. Otherwise, they are different.

- Under the BSM model, we can also assume that there exists such an artificial risk-neutral world, in which the expected returns on all assets earn risk-free rate.

- The stock price dynamics under the risk-neutral world becomes,
  \[ \frac{dS_t}{S_t} = (r - q)dt + \sigma dW_t. \]

- Simply replace the actual expected return \( \mu \) with the return from a risk-neutral world \( (r - q) \) \([ex-dividend return]\).
The risk-neutral return on spots

\[ \frac{dS_t}{S_t} = (r - q)dt + \sigma dW_t, \] under risk-neutral probabilities.

- In the risk-neutral world, investing in all securities make the riskfree rate as the total return.
- If a stock pays a dividend yield of \( q \), then the risk-neutral expected return from stock price appreciation is \( r - q \), such as the total expected return is: dividend yield + price appreciation = \( r \).
- Investing in a currency earns the foreign interest rate \( r_f \) similar to dividend yield. Hence, the risk-neutral expected currency appreciation is \( r - r_f \) so that the total expected return is still \( r \).
- Regard \( q \) as \( r_f \) and value options as if they are the same.
The risk-neutral return on forwards/futures

- If we sign a forward contract, we do not pay anything upfront and we do not receive anything in the middle (no dividends or foreign interest rates). Any P&L at expiry is excess return.

- Under the risk-neutral world, we do not make any excess return. Hence, the forward price dynamics has zero mean (driftless) under the risk-neutral probabilities: \( \frac{dF_t}{F_t} = \sigma dW_t \).

- The carrying costs are all hidden under the forward price, making the pricing equations simpler.
Readings behind the technical jargons: $\mathbb{P}$ v. $\mathbb{Q}$

- $\mathbb{P}$: Actual probabilities that cashflows will be high or low, estimated based on historical data and other insights about the company.
  
  Valuation is all about getting the forecasts right and assigning the appropriate price for the forecasted risk — *fair wrt future cashflows and your risk preference*.

- $\mathbb{Q}$: “Risk-neutral” probabilities that we can use to aggregate expected future payoffs and discount them back with riskfree rate, regardless of how risky the cash flow is.
  
  - It is related to real-time scenarios, but it has nothing to do with real-time probability.
  - Since the intention is to hedge away risk under all scenarios and discount back with riskfree rate, we do not really care about the actual probability of each scenario happening. We just care about what all the possible scenarios are and whether our hedging works under all scenarios.
  - $\mathbb{Q}$ is not about getting close to the actual probability, but about being *fair wrt the prices of securities that you use for hedging*. 

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The BSM Delta

The BSM delta of European options (Can you derive them?):

\[ \Delta_c \equiv \frac{\partial c_t}{\partial S_t} = e^{-q\tau} N(d_1), \quad \Delta_p \equiv \frac{\partial p_t}{\partial S_t} = -e^{-q\tau} N(-d_1) \]

\( (S_t = 100, \ T - t = 1, \sigma = 20\%) \)

- Industry quotes the delta in absolute percentage terms (right panel).
- Which of the following is out-of-the-money? (i) 25-delta call, (ii) 25-delta put, (iii) 75-delta call, (iv) 75-delta put.
- The strike of a 25-delta call is close to the strike of: (i) 25-delta put, (ii) 50-delta put, (iii) 75-delta put.
Delta as a moneyness measure

Different ways of measuring moneyness:

- $K$ (relative to $S$ or $F$): Raw measure, not comparable across different stocks.
- $K/F$: better scaling than $K - F$.
- $\ln(K/F)$: more symmetric under BSM.
- $\frac{\ln(K/F)}{\sigma \sqrt{(T-t)}}$: standardized by volatility and option maturity, comparable across stocks. Need to decide what $\sigma$ to use (ATMV, IV, 1).
- $d_1$: a standardized variable.
- $d_2$: Under BSM, this variable is the truly standardized normal variable with $\phi(0, 1)$ under the risk-neutral measure.
- delta: Used frequently in the industry, quoted in absolute percentages.
  - Measures moneyness: Approximately the percentage chance the option will be in the money at expiry.
  - Reveals your underlying exposure (how many shares needed to achieve delta-neutral).
Delta hedging

Example: A bank has sold for $300,000 a European call option on 100,000 shares of a nondividend paying stock, with the following information: $S_t = 49, K = 50, r = 5\%, \sigma = 20\%, (T - t) = 20\text{ weeks}, \mu = 13\%$.

- What’s the BSM value for the option? $\rightarrow$ $2.4$
- What’s the BSM delta for the option? $\rightarrow$ 0.5216.

Strategies:

- **Naked position**: Take no position in the underlying.
- **Covered position**: Buy 100,000 shares of the underlying.
- **Stop-loss strategy**: Buy 100,000 shares as soon as price reaches $50, sell 100,000 shares as soon as price falls below $50.
- **Delta hedging**: Buy 52,000 share of the underlying stock now. Adjust the shares over time to maintain a *delta-neutral portfolio*.
  - Need frequent rebalancing (daily) to maintain delta neutral.
  - Involves a “buy high and sell low” trading rule.
Delta hedging with futures

- The delta of a futures contract is \( e^{(r-q)(T-t)} \).
- The delta of the option with respect to (wrt) futures is the delta of the option over the delta of the futures.
- The delta of the option wrt futures (of the same maturity) is
  \[
  \Delta_{c/F} \equiv \frac{\partial c_t}{\partial F_{t,T}} = \frac{\partial c_t/\partial S_t}{\partial F_{t,T}/\partial S_t} = e^{-r\tau} N(d_1),
  \]
  \[
  \Delta_{p/F} \equiv \frac{\partial p_t}{\partial F_{t,T}} = \frac{\partial p_t/\partial S_t}{\partial F_{t,T}/\partial S_t} = -e^{-r\tau} N(-d_1).
  \]
- Whenever available (such as on indexes, commodities), using futures to delta hedge can potentially reduce transaction costs.
OTC quoting and trading conventions for currency options

- Options are quoted at fixed time-to-maturity (not fixed expiry date).
- Options at each maturity are not quoted in invoice prices (dollars), but in the following format:
  - Delta-neutral straddle implied volatility (ATMV):
    A straddle is a portfolio of a call & a put at the same strike. The strike here is set to make the portfolio delta-neutral \( d_1 = 0 \).
  - 25-delta risk reversal: \( RR_{25} = IV(\Delta_c = 25) - IV(\Delta_p = 25) \).
  - 25-delta butterfly spreads:
    \( BF_{25} = (IV(\Delta_c = 25) + IV(\Delta_p = 25))/2 - ATMV \).
  - Risk reversals and butterfly spreads at other deltas, e.g., 10-delta.

- When trading, invoice prices and strikes are calculated based on the BSM formula.
- The two parties exchange both the option and the underlying delta.
  - The trades are delta-neutral.
The BSM vega

- Vega ($\nu$) is the rate of change of the value of a derivatives portfolio with respect to volatility — it is a measure of the volatility exposure.

- BSM vega: the same for call and put options of the same maturity

$$\nu \equiv \frac{\partial c_t}{\partial \sigma} = \frac{\partial p_t}{\partial \sigma} = S_t e^{-q(T-t)} \sqrt{T-t} n(d_1)$$

$n(d_1)$ is the standard normal probability density: $n(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}$.

$(S_t = 100, T - t = 1, \sigma = 20\%)$

Volatility exposure (vega) is higher for at-the-money options.
Vega hedging

- Delta can be changed by taking a position in the underlying.
- To adjust the volatility exposure (vega), it is necessary to take a position in an option or other derivatives.

Hedging in practice:
- Traders usually ensure that their portfolios are delta-neutral at least once a day.
- Whenever the opportunity arises, they improve/manage their vega exposure — options trading is more expensive.
- As portfolio becomes larger, hedging becomes less expensive.

- Under the assumption of BSM, vega hedging is not necessary: $\sigma$ does not change. But in reality, it does.
  - Vega hedge is outside the BSM model.
Example: Delta and vega hedging

Consider an option portfolio that is delta-neutral but with a vega of $-8,000$. We plan to make the portfolio both delta and vega neutral using two instruments:

- The underlying stock
- A traded option with delta 0.6 and vega 2.0.

How many shares of the underlying stock and the traded option contracts do we need?

- To achieve vega neutral, we need long $8000/2 = 4,000$ contracts of the traded option.
- With the traded option added to the portfolio, the delta of the portfolio increases from 0 to $0.6 \times 4,000 = 2,400$.
- We hence also need to short 2,400 shares of the underlying stock $\Rightarrow$ each share of the stock has a delta of one.
Another example: Delta and vega hedging

Consider an option portfolio with a delta of 2,000 and vega of 60,000. We plan to make the portfolio both delta and vega neutral using:

- The underlying stock
- A traded option with delta 0.5 and vega 10.

*How many shares of the underlying stock and the traded option contracts do we need?*

- As before, it is easier to take care of the vega first and then worry about the delta using stocks.
- To achieve vega neutral, we need short/write \( \frac{60000}{10} = 6000 \) contracts of the traded option.
- With the traded option position added to the portfolio, the delta of the portfolio becomes \( 2000 - 0.5 \times 6000 = -1000 \).
- We hence also need to long 1000 shares of the underlying stock.
A more formal setup*

Let \((\Delta_p, \Delta_1, \Delta_2)\) denote the delta of the existing portfolio and the two hedging instruments. Let \((\nu_p, \nu_1, \nu_2)\) denote their vega. Let \((n_1, n_2)\) denote the shares of the two instruments needed to achieve the target delta and vega exposure \((\Delta_T, \nu_T)\). We have

\[
\begin{align*}
\Delta_T &= \Delta_p + n_1 \Delta_1 + n_2 \Delta_2 \\
\nu_T &= \nu_p + n_1 \nu_1 + n_2 \nu_2
\end{align*}
\]

We can solve the two unknowns \((n_1, n_2)\) from the two equations.

- **Example 1:** The stock has delta of 1 and zero vega.
  
  \[
  \begin{align*}
  0 &= 0 + n_1 0.6 + n_2 \\
  0 &= -8000 + n_1 2 + 0
  \end{align*}
  \]
  
  \[n_1 = 4000, \quad n_2 = -0.6 \times 4000 = -2400.\]

- **Example 2:** The stock has delta of 1 and zero vega.

  \[
  \begin{align*}
  0 &= 2000 + n_1 0.5 + n_2 \\
  0 &= 60000 + n_1 10 + 0
  \end{align*}
  \]

\[n_1 = -6000, \quad n_2 = 1000.\]

- **When do you want to have non-zero target exposures?**
BSM gamma

- Gamma ($\Gamma$) is the rate of change of delta ($\Delta$) with respect to the price of the underlying asset.

- The BSM gamma is the same for calls and puts:

$$\Gamma \equiv \frac{\partial^2 c_t}{\partial S_t^2} = \frac{\partial \Delta_t}{\partial S_t} = \frac{e^{-q(T-t)}n(d_1)}{S_t \sigma \sqrt{T-t}}$$

\[(S_t = 100, T-t = 1, \sigma = 20\%)

Gamma is high for near-the-money options. High gamma implies high variation in delta, and hence more frequent rebalancing to maintain low delta exposure.
Gamma hedging

- High gamma implies high variation in delta, and hence more frequent rebalancing to maintain low delta exposure.

- Delta hedging is based on small moves during a very short time period.
  - assuming that the relation between option and the stock is linear locally.

- When gamma is high,
  - The relation is more curved (convex) than linear,
  - The P&L (hedging error) is more likely to be large in the presence of large moves.

- The gamma of a stock is zero.

- We can use traded options to adjust the gamma of a portfolio, similar to what we have done to vega.

- But if we are really concerned about large moves, we may want to try something else.
Static hedging

- Dynamic hedging can generate large hedging errors when the underlying variable (stock price) can jump randomly.
  - A large move size per se is not an issue, as long as we know how much it moves — a binomial tree can be very large moves, but delta hedge works perfectly.
  - As long as we know the magnitude, hedging is relatively easy.
  - The key problem comes from large moves of random size.

- An alternative is to devise static hedging strategies: The position of the hedging instruments does not vary over time.
  - Conceptually not as easy. Different derivative products ask for different static strategies.
  - It involves more option positions. Cost per transaction is high.
  - Monitoring cost is low. Fewer transactions.

- Examples:
  - “Static hedging of standard options,” Carr and Wu, JFEC.
  - “Simple Robust Hedging With Nearby Contracts,” Wu and Zhu, working paper.
Summary

- Understand the basic properties of normally distributed random variables.
- Map a stochastic process to a random variable.
- Understand the link between BSM and the binomial model.
- Memorize the BSM formula (any version).
- Understand forward pricing and link option pricing to forward pricing.
- Understand the risk exposures measured by delta, vega, and gamma.
- Form portfolios to neutralize delta and vega risk.