Modeling Credit Risk

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NUS-Santander Doctorate Workshop in Advanced Financial Risk Management
July 15-19, 2013
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Notations and terminologies

- $B(t, T)$ — the time-$t$ value of a zero-coupon bond that pays $1$ at time $T$.

- $y(t, T)$ — the time-$t$ continuously compounded spot interest rate with expiry rate $T$.

  
  \[ y(t, T) = -\frac{\ln B(t, T)}{T - t}, \quad B(t, T) = \exp(-y(t, T)(T - t)), \quad B(t, t) = 1. \]

- $f(t, T, S)$ — The time-$t$ continuously compounded forward rate prevailing at future time interval $(T, S)$:

  \[ f(t, T, S) = -\frac{\ln B(t, S)/B(t, T)}{S - T}, \quad \frac{B(t, S)}{B(t, T)} = e^{-f(t, T, S)(S - T)} \]

- $f(t, T)$ — the time-$t$ instantaneous forward rate prevailing at $T$:

  \[ f(t, T) = f(t, T, T) = -\frac{\partial \ln B(t, T)}{\partial T}, \quad B(t, T) = e^{-\int_t^T f(t, s) \, ds} \]

- $r(t)$ — the instantaneous interest rate defined by the limit:

  \[ r(t) = f(t, t) = \lim_{T \downarrow t} y(t, T). \]
The pricing kernel

- If there is no arbitrage in a market, there must exist at least one strictly positive process $M_t$ such that the deflated gains process associated with any trading strategy is a martingale:

$$M_t P_t = \mathbb{E}_t^P [M_T P_T], \quad (1)$$

where $P_t$ denotes the time-$t$ value of a trading strategy, $\mathbb{P}$ denotes the statistical probability measure.

- Consider the strategy of buying a default-free zero-coupon bond at time $t$ and hold it to maturity at $T$, we have

$$M_t B(t, T) = \mathbb{E}_t^P [M_T B(T, T)] = \mathbb{E}_t^P [M_T].$$

- If the market is complete, this process $M_t$ is unique.

- If the available securities cannot complete the market, there can be multiple processes that satisfy equation (1).

- If we cannot find a single positive process that can price all securities, there is arbitrage.

- $M_t$ is called the state-price deflator. The ratio $M_{t,T} = M_T / M_t$ is called the stochastic discount factor, or the pricing kernel: $B(t, T) = \mathbb{E}_t^P [M_{t,T}].$
From pricing kernel to exchange rates

Let $M_{t,T}^h$ denote the pricing kernel in economy $h$ that prices all securities in that economy with its currency denomination.

The $h$-currency price of currency-$f$ ($h$ is home currency) is linked to the pricing kernels of the two economies by,

$$\frac{S_{T}^{fh}}{S_{t}^{fh}} = \frac{M_{t,T}^f}{M_{t,T}^h}$$

The log currency return over period $[t, T]$, $\ln S_{T}^{fh}/S_{t}^{fh}$ equals the difference between the log pricing kernels of the two economies.

Let $S$ denote the dollar price of pound (e.g. $S_t = $2.06), then $\ln S_{T}/S_{t} = \ln M_{t,T}^{pound} - \ln M_{t,T}^{dollar}$. 

If markets are completed by primary securities (e.g., bonds and stocks), there is one unique pricing kernel per economy. The exchange rate movement is uniquely determined by the ratio of the pricing kernels.

If the markets are not completed by primary securities, exchange rates (and currency options) help complete the markets by requiring that the ratio of any two viable pricing kernels go through the exchange rate.
Multiplicative decomposition of the pricing kernel

- In a discrete-time representative agent economy with additive CRRA utility, the pricing kernel equals the ratio of the marginal utilities of consumption,

\[ M_{t,t+1} = \beta \frac{u'(c_{t+1})}{u'(c_t)} = \beta \left( \frac{c_{t+1}}{c_t} \right)^{-\gamma}, \quad \gamma \text{ - relative risk aversion.} \]

- In continuous time, it is convenient to perform the following multiplicative decomposition on the pricing kernel:

\[ M_{t,T} = \exp \left( -\int_t^T r_s ds \right) \mathcal{E} \left( -\int_t^T \gamma_s^s dX_s \right) \]

where

- \( r \) is the instantaneous riskfree rate,
- \( \mathcal{E} \) is the stochastic exponential martingale operator,
- \( X \) denotes the risk sources in the economy, and
- \( \gamma \) is the market price of the risk \( X \).

- If \( X_t = W_t \), \( \mathcal{E} \left( -\int_t^T \gamma_s dW_s \right) = e^{-\int_t^T \gamma_s dW_s - \frac{1}{2} \int_t^T \gamma_s^2 ds} \).

- In a continuous time version of the representative agent example, \( dX_s = d\ln c_t \) and \( \gamma \) is relative risk aversion.

- \( r \) is normally a function of \( X \).
Bond pricing

- Recall the multiplicative decomposition of the pricing kernel:

$$M_{t,T} = \exp \left( - \int_t^T r_s ds \right) \mathcal{E} \left( - \int_t^T \gamma_s^\top dX_s \right)$$

- Given the pricing kernel, the value of the zero-coupon bond can be written as

$$B(t, T) = \mathbb{E}_t^P [M_{t,T}] = \mathbb{E}_t^P \left[ \exp \left( - \int_t^T r_s ds \right) \mathcal{E} \left( - \int_t^T \gamma_s^\top dX_s \right) \right] = \mathbb{E}_t^Q \left[ \exp \left( - \int_t^T r_s ds \right) \right]$$

where the measure change from $\mathbb{P}$ to $\mathbb{Q}$ is defined by the exponential martingale $\mathcal{E}(\cdot)$.

- The risk sources $X$ and their market prices $\gamma$ matter for bond pricing through the correlation between $r$ and $X$. 
Consider the following exponential martingale that defines the measure change from $\mathbb{P}$ to $\mathbb{Q}$:

\[
\frac{d\mathbb{Q}}{d\mathbb{P}}|_t = \mathcal{E} \left( -\int_0^t \gamma_s dW_s \right),
\]

- If the $\mathbb{P}$ dynamics is: $dS^i_t = \mu^i S^i_t dt + \sigma^i S^i_t dW^i_t$, with $\rho^i dt = \mathbb{E}[dW_t dW^i_t]$, then the dynamics of $S^i_t$ under $\mathbb{Q}$ is: $dS^i_t = \mu^i S^i_t dt + \sigma^i S^i_t dW^i_t + \mathbb{E}[-\gamma_t dW_t \sigma^i S^i_t dW^i_t] = (\mu^i_t - \gamma \sigma^i \rho^i) S^i_t dt + \sigma^i S^i_t dW^i_t$

- If $S^i_t$ is the price of a traded security, we need $r = \mu^i_t - \gamma \sigma^i \rho^i$. The risk premium on the security is $\mu^i_t - r = \gamma \sigma^i \rho^i$.

How do things change if the pricing kernel is given by:

\[
M_{t,T} = \exp \left( -\int_t^T r_s ds \right) \mathcal{E} \left( -\int_t^T \gamma_s^T dW_s \right) \mathcal{E} \left( -\int_t^T \gamma_s^T dZ_s \right),
\]

where $Z_t$ is another set of Brownian motions independent of $W_t$ or $W^i_t$.

- $Z$ does not affect bond pricing if it is not correlated with $r$.

Comment: Bond price represents a projection of the pricing kernel: Not all things show up in a projection.
Let $X$ denote a pure jump process with its compensator being $\nu(x, t)$ under $\mathbb{P}$.

Consider a measure change defined by the exponential martingale:
$$\frac{dQ}{dP} \bigg|_t = \mathcal{E}(-\gamma X_t),$$

The compensator of the jump process under $Q$ becomes:
$$\nu(x, t)^Q = e^{-\gamma x} \nu(x, t).$$

Example: Merton (1976)'s compound Poisson jump process,
$$\nu(x, t) = \lambda \frac{1}{\sigma_J \sqrt{2\pi}} e^{-\frac{(x-\mu_J)^2}{2\sigma_J^2}}.$$ Under $Q$, it becomes
$$\nu(x, t)^Q = \lambda \frac{1}{\sigma_J \sqrt{2\pi}} e^{-\gamma x - \frac{(x-\mu_J)^2}{2\sigma_J^2}} = \lambda^Q \frac{1}{\sigma_J \sqrt{2\pi}} e^{-\frac{(x-\mu_J^Q)^2}{2\sigma_J^2}}$$
with $\mu^Q = \mu_J - \gamma \sigma_J^2$ and $\lambda^Q = \lambda e^{\frac{1}{2} \gamma (\gamma \sigma_J^2 - 2\mu_J)}$.

Dynamic term structure models

A long list of papers propose different dynamic term structure models:

- **Specific examples:**
  - Vasicek, 1977, JFE: The instantaneous interest rate follows an Ornstein-Uhlenbeck process.
  - Cox, Ingersoll, Ross, 1985, Econometrica: The instantaneous interest rate follows a square-root process.
  - Many multi-factor examples ...

- **Classifications (back-filling)**
  - Duffie, Kan, 1996, Mathematical Finance: Spot rates are affine functions of state variables.
  - Leippold, Wu, 2002, JFQA: Spot rates are quadratic functions of state variables.
  - Filipovic, 2002, Mathematical Finance: How far can we go?
The “forward-looking” decomposition of the term structure

- Interest rates can be different at different terms (maturities). The pattern across different terms is referred to as the *term structure behavior*.

- The “forward-looking” approach of term structure modeling: Start with the short rate \((r)\) dynamics, the market prices of the short rate risk \((\gamma)\), based on what we think is reasonable; and derive the fair valuation of bonds based on these dynamics and market price specifications.

- Generically speaking, the interest rate term structure can be decomposed into three components:
  2. **Risk premium**: Long-term bonds are more sensitive to rate movements (and riskier), and hence higher returns (yields) on average.
  3. **Convexity**: The bond value dependence on rate is convex, more so for long-term bonds. High rate volatility benefits long bond holders. As such, they can afford to ask for a lower expected return (yield).

- The decomposition can be most clearly seen from the simplified Vasick model: \(dr_t = \sigma dW_t, \quad f(t, \tau) = r_t - \gamma \sigma \tau - \frac{1}{2} \sigma^2 \tau^2\).
The “back-filling” procedure of DTSM identification

- The “forward-looking” procedure:
  1. Start by making assumptions on factor dynamics \(X\), market prices \(\gamma\), and how interest rates are related to the factors \(r(X)\), based on what we think is reasonable.
  2. Derive the fair valuation of bonds based on these dynamics and market price specifications.

- The back-filling (reverse engineering) procedure:
  1. State the form of solution that we desire for bond prices (spot rates).
  2. Figure out what dynamics specifications generate the pricing solutions that we desire.
    - The dynamics are not specified to be reasonable, but specified to generate a form of solution that we like.

- It is good to be able to go both ways.
  - The back-filling procedures tells us what’s tractable, the traditional procedure puts the appropriate focus on economic sense.
Back-filling affine models

- Target: Zero-coupon bond prices are exponential affine functions of factors.
  - Continuously compounded spot rates are affine in factors.
  - It is simple and tractable. We can use spot rates as factors.
- Let $X$ denote the state variables, let $B(X_t, \tau)$ denote the time-$t$ fair value of a zero-coupon bond with time to maturity $\tau = T - t$, we have
  \[
  B(X_t, \tau) = \mathbb{E}_t^Q \left[ \exp \left( - \int_t^\tau r(X_s) \, ds \right) \right] = \exp (-a(\tau) - b(\tau)^\top X_t)
  \]
- Implicit assumptions:
  - By writing $B(X_t, \tau)$ and $r(X_t)$, and solutions $A(\tau), b(\tau)$, I am implicitly focusing on time-homogeneous models. Calendar dates do not matter. This assumption is for (notational) simplicity more than anything else.
  - With calendar time dependence, the notation can be changed to, $B(X_t, t, T)$ and $r(X_t, t)$. The solutions would be $a(t, T), b(t, T)$.
- Questions to be answered:
  - What is the short rate function $r(X_t)$?
  - What’s the dynamics of $X_t$ under measure $\mathbb{Q}$?
Diffusion dynamics

- To make the derivation easier, let’s focus on diffusion factor dynamics:
  \[ dX_t = \mu(X)dt + \sigma(X)dW_t \] under \( \mathbb{Q} \).

- We want to know: What kind of specifications for \( \mu(X), \sigma(X) \) and \( r(X) \) generate the affine solutions?

- For a generic valuation problem,
  \[ f(X_t, t, T) = \mathbb{E}_t^{\mathbb{Q}} \left[ \exp \left( - \int_t^T r(X_s)ds \right) \Pi_T \right], \]
  where \( \Pi_T \) denotes terminal payoff, the value satisfies the following partial differential equation:
  \[ f_t + \mathcal{L}f = rf, \quad \mathcal{L}f \text{ – infinitesimal generator} \]
  with boundary condition \( f(T) = \Pi_T \).

- Apply the PDE to the bond valuation problem,
  \[ B_t + B_X^\top \mu(X) + \frac{1}{2} \sum B_{XX} \cdot \sigma(X)\sigma(X)^\top = rB \]
  with boundary condition \( B(X_T, 0) = 1 \).
Starting with the PDE,

\[ B_t + B_X^\top \mu(X) + \frac{1}{2} \sum B_{XX} \cdot \sigma(X)\sigma(X)^\top = rB, \quad B(X_T, 0) = 1. \]

If \( B(X_t, \tau) = \exp(-a(\tau) - b(\tau)^\top X_t) \), we have

\[ B_t = B \left( a'(\tau) + b'(\tau)^\top X_t \right), \quad B_X = -Bb(\tau), \quad B_{XX} = Bb(\tau)b(\tau)^\top, \]
\[ y(t, \tau) = \frac{1}{\tau} \left( a(\tau) + b(\tau)^\top X_t \right), \quad r(X_t) = a'(0) + b'(0)^\top X_t = a_r + b_r^\top X_t. \]

Plug these back to the PDE,

\[ a'(\tau) + b'(\tau)^\top X_t - b(\tau)^\top \mu(X) + \frac{1}{2} \sum b(\tau)b(\tau)^\top \cdot \sigma(X)\sigma(X)^\top = a_r + b_r^\top X_t \]

Question: What specifications of \( \mu(X) \) and \( \sigma(X) \) guarantee the above PDE to hold at all \( X \)?

- Power expand \( \mu(X) \) and \( \sigma(X)\sigma(X)^\top \) around \( X \) and then collect coefficients of \( X^p \) for \( p = 0, 1, 2, \ldots \). These coefficients have to be zero separately for the PDE to hold at all times.
Back filling

\[ a'(\tau) + b'(\tau)^T X_t - b(\tau)^T \mu(X) + \frac{1}{2} \sum b(\tau)b(\tau)^T \cdot \sigma(X)\sigma(X)^T = a_r + b_r^T X_t \]

Set \( \mu(X) = a_m + b_m X + c_m XX^T + \cdots \) and
\[ [\sigma(X)\sigma(X)^T]_i = \alpha_i + \beta_i^T X + \eta_i XX^T + \cdots, \]
and collect terms:

- The quadratic and higher-order terms are almost surely zero.
- We thus have the conditions to have exponential-affine bond prices:

\[ \mu(X) = a_m + b_m X, \quad [\sigma(X)\sigma(X)^T]_i = \alpha_i + \beta_i^T X, \quad r(X) = a_r + b_r^T X. \]

- We can solve the coefficients \([a(\tau), b(\tau)]\) via the following ordinary differential equations:

\[ a'(\tau) = a_r + b(\tau)^T a_m - \frac{1}{2} \sum b(\tau)b(\tau)^T \cdot \alpha_i \]
\[ b'(\tau) = b_r + b_m^T b(\tau) - \frac{1}{2} \sum b(\tau)b(\tau)^T \cdot \beta_i \]

starting at \( a(0) = 0 \) and \( b(0) = 0 \).
Homework

1. Work out the full-version Vasicek model, understand the contribution of each component (expectation, risk premium, convexity) to different segments of the term structure.

2. Try the back-filling procedure on a new class of models:
   - Suppose we want all rates to take values between zero and infinity. — it is a reasonable assumption.
   - One way to guarantee this is to assume that all rates are quadratic functions of the factors:
     \[ y(X_t, \tau) = X_t^\top C(\tau)X_t \]
     with \( C(\tau) \) being a positive definite matrix.
   - Question: What kind of diffusion risk-neutral dynamics for \( X \) can guarantee this nice solution?
     - You can also work with processes with jumps, but working with pure diffusion dynamics is a lot easier...
From $\mathbb{Q}$ to $\mathbb{P}$, not from $\mathbb{P}$ to $\mathbb{Q}$

- The affine conditions are on the $\mathbb{Q}$-dynamics, not $\mathbb{P}$-dynamics.

- Traditionally, researchers start with the $\mathbb{P}$-dynamics (the real thing), and specify risk preferences (market prices of risks). From these two, they derive the $\mathbb{Q}$-dynamics and then valuation. —This way of thinking is natural economically.

- Yet, the back-filling exercise shows that the real requirement for tractability (e.g., affine) is on $\mathbb{Q}$-dynamics.

- If *tractability* is the major concern, one should reverse the way of thinking and start with $\mathbb{Q}$-dynamics to generate tractable derivative pricing solutions. — Fill in *any* market price of risk specification: The only constraints are reasonability and identification, not tractability.

- The reverse thinking can also be beneficial when it is difficult to identify $\mathbb{P}$-dynamics accurately from a single time series.
  - Identify $\mathbb{Q}$-dynamics first with (lots of) derivative price observations.
  - Use a simple market price of risk specification to transfer the estimated $\mathbb{Q}$-dynamics to $\mathbb{P}$.
  - Caveat: If the derivative prices are wrong/noisy, the identified $\mathbb{Q}$ and then $\mathbb{P}$ can also be affected negatively.
A one-factor example

- Suppose the short rate follows the following \( \mathbb{P} \)-dynamics
  \[
  dr_t = \left( a + br_t + cr_t^2 + dr_t^3 + er_t^4 \right) dt + \sigma_r dW_t
  \]
  and we specify the martingale component of the pricing kernel as
  \[
  \mathcal{E} \left( -\int_t^T \gamma_s dW_t \right), \text{ with } \gamma_s = \gamma_0 + \gamma_1 r_t + (c/\sigma_r)r_t^2 + (d/\sigma_r)r_t^3 + (d/\sigma_r)r_t^4.
  \]
- Can you price zero-coupon bonds based on this specification?
- Reversely, if the \( \mathbb{Q} \)-dynamics are given by
  \[
  \mu(X) = a_m + b_m X, \quad [\sigma(X)\sigma(X)^\top]_i = \alpha_i + \beta_i^\top X
  \]
  and the measure change is defined by \( \mathcal{E} \left( -\int_t^T \gamma_s^\top dW_t \right) \), we have the drift of the \( \mathbb{P} \)-dynamics as
  \[
  \mu(X)_{\mathbb{P}} = a_m + b_m X + \sigma(X)\gamma_t
  \]
  The market price of risk \( \gamma_t \) can be anything...
- Identification of complicated market price of risk is the major concern...
Pricing defaultable bonds

- For a default-free zero-coupon bond, the terminal payoff is certain at $\Pi_T = 1$. The valuation is
  
  \[ B(t, T) = \mathbb{E}_t [M_{t,T}] = \mathbb{E}_t^Q \left[ \exp \left( - \int_t^T r_s ds \right) \right]. \]

- If the bond can default, Duffie and Singleton (1999) show that for pricing purposes, one can replace the discount rate $r_t$ with the “default-adjusted” rate $r_t + h_t L_t$, where $h$ is the risk-neutral intensity of default and $L$ is risk-neutral expected fractional loss in market value in the event of default.
  - The application is more general than pricing zero-coupon bonds.
  - When it is difficult to distinguish the two components $h_t$ and $L_t$, one can directly model the “instantaneous spread” $s_t = h_t L_t$.
  - Liquidity effects can be analogously captured by an instantaneous spread (Duffie, Pedersen, and Singleton (2003)).
  - One can also directly model a default and liquidity-adjusted instantaneous rate $r_t$.
  - How one should decompose the discount rate is mainly driven by identification and the question one wants to address.
Examples

1. If one wants to model the interaction between on-the-run and off-the-run Treasuries, one can start with an instantaneous interest rate \( r \) for on-the-run bonds, and add an instantaneous spread \( s_t \) for off-the-run bonds.

2. Corporate bond pricing:
   - One normally uses the on-the-run Treasury to define benchmark “risk free” rate curve, even though nowadays Treasuries are no longer regarded as risk free.
   - The effects of credit (and other) risks in a corporate bond can be captured by an instantaneous spread on top of the Treasury curve.

3. Pricing equity options, credit default swap (CDS) spreads ...
   - Traditionally researchers use the Treasury rates to define the “riskfree” curve for option pricing, now most (including OptionMetrics) has switched to the underlying LIBOR/swap curve, which is not risk free.
   - For CDS and other credit risk modeling, the instantaneous credit (and liquidity) spread \( s_t \) is also defined on top of the LIBOR/swap curve.
   - Instead of calling \( r_t \) the risk free rate, I often refer to it as the ”benchmark” rate.
Pricing CDS as an example

- A CDS is an OTC contract between the seller and the buyer of protection against the risk of default on a set of debt obligations issued by a reference entity.
- It is essentially an insurance policy that protects the buyer against the loss of principal on a bond in case of a default by the issuer.
- The protection buyer pays a periodic premium over the life of the contract and is, in turn, covered for the period.
- The premium paid by the protection buyer to the seller is often termed as the “CDS spread” and is quoted in basis points per annum of the contract’s notional value and is usually paid quarterly.
- If a certain pre-specified credit event occurs, the premium payment stops and the protection seller pays the buyer the par value for the bond.
- If no credit event occurs during the term of the swap, the protection buyer continues to pay the premium until maturity.
- The contract started in the sovereign market in mid 90s, but the volume has moved to corporate entities.
Credit events

- A CDS is triggered if, during the term of protection, an event that materially affects the cashflows of the reference debt obligation takes place.
- A credit event can be a bankruptcy of the reference entity, or a default of a bond or other debt issued by the reference entity.
- Restructuring is considered a credit event for some, but not all, CDS contracts, referred to as “R”, “mod-R”, or “modmod- R”. Events such as principal/interest rate reduction/deferral and changes in priority ranking, currency, or composition of payment can qualify as credit events.
- When a credit event triggers the CDS, the contract is settled and terminated. The settlement can be physical or cash. The protection buyer has a right to deliver any deliverable debt obligation of the reference entity to the protection seller in exchange for par.
- There can be additional maturity restrictions if the triggering credit event is a restructuring.
- The CDS buyer and the seller can also agree to cash settle the contract at the time of inception or exercise. In this case, the protection seller pays an amount equal to par less the market value of a deliverable obligation.
A schematic chart of the cashflows

CDS Cashflows before Maturity/Default

- Protection Buyer
- Quarterly Premium
- Protection Seller
- Protection on Default

Physical Settlement in Case of Default

- Protection Buyer
- Deliverable Obligation
- Protection Seller
- Par

Cash Settlement in Case of Default

- Protection Buyer
- Par – Recovery Value
- Protection Seller
A CDS contract specifies the precise name of the legal entity on which it provides default protection.

Given the possibility of existence of several legal entities associated with a company, a default by one of them may not be tantamount to a default on the CDS. —It is important to know the exact name of the legal entity and the seniority of the capital structure covered by the CDS.

Changes in ownership of the reference entity's bonds or loans can also result in a change in the reference entity covered by the CDS contract.

<table>
<thead>
<tr>
<th>Ownership of bonds/loans</th>
<th>New reference entity</th>
</tr>
</thead>
<tbody>
<tr>
<td>One entity assumes more than 75%</td>
<td>Successor</td>
</tr>
<tr>
<td>No entity assumes more than 75%, but one of more entities assume 25-75%</td>
<td>Divide the contract equally among such entities</td>
</tr>
<tr>
<td>No entity assumes more than 25%</td>
<td>Original legal entity</td>
</tr>
</tbody>
</table>

If the legal entity does not survive, the CDS contract follows the entity that succeeds to highest percentage of bonds or loans.
Since 2002, a vast majority of CDS contracts have standardized quarterly payment and maturity dates — 20th of March, June, September, and December.

This standardization has several benefits including convenience in offsetting CDS trades, rolling over of contracts, relative value trading, single name vs. the benchmark indices or tranched index products trading.

After the recent financial crises, there is a strong movement toward central clearing and more standardization:

- The CDS spread is standardized to either 100 and 500 basis points.
- The value of the contract is settled with upfront payment.
The risk profile of a CDS is similar to that of a corporate bond of the reference entity, but with several important differences.

- A CDS does not require an initial funding, which allows leveraged positions.
- A CDS transaction can be entered where a cash bond of the reference entity at a particular maturity is not available.
- By entering a CDS contract as a protection seller, an investor can easily create a short position in the reference credit.

Most contracts fall between $10 million to $20 million in notional amount. Maturity ranges from one to ten years, with the five-year maturity being the most common.
CDS pricing

- The fair CDS spread (premium) is set to equate the present value of all premium payments to the present value of the expected default loss, both in risk-adjusted sense.

\[ \text{Present value of premium payment} = \frac{N(1-p)s}{1+r} \]

\[ \text{Present value of expected default loss} = \frac{Np(1-R)}{1+r} \]

Equating the values of the two legs, we have

\[ s = \frac{p(1-R)}{1-p} \approx p(1-R) \]

Or from a CDS quote \( s \), we can learn the risk-adjusted default probability as

\[ p = \frac{s}{s+1-R} \approx \frac{s}{1-R} \]

- There is typically an accrued premium when default does not happen exactly on the quarterly payment dates.

- Consider a one period toy example, in which the premium \( s \) is paid at the end of the period and default can only happen at the end of the period with risk-adjusted probability \( p \). In case of default, the bond’s recovery rate is \( R \).
  - The present value of the premium payment is: \( N(1-p)s/(1+r) \).
  - The present value of the expected default loss: \( Np(1-R)/(1+r) \).
  - Equating the values of the two legs, we have \( s = \frac{p(1-R)}{1-p} \approx p(1-R) \).
  - Or from a CDS quote \( s \), we can learn the risk-adjusted default probability as \( p = \frac{s}{s+1-R} \approx \frac{s}{1-R} \).
CDS pricing: A continuous time setup

- Let $r_t$ denote the instantaneous benchmark interest rate.
- Let $s$ denote the annual premium rate paid continuously until default.
- Default arrives unexpectedly in a Cox process with arrival rate $\lambda_t$.
  - The probability that a default will occur in the small time interval $[t, t + \Delta t]$ is approximately $\lambda_t \Delta t$.
  - The probability that the entity survives up to $t$ is $S(t) = e^{-\int_0^t \lambda_s ds}$.
  - Default probability is $Q(t) = 1 - S(t)$.
- The value of the premium leg is: $\mathbb{E}_0 \left[ s_0 \int_0^T e^{-\int_0^t (r_s + \lambda_s) ds} dt \right]$.
- The value of the protection leg is: $\mathbb{E}_0 \left[ (1 - R) \int_0^T \lambda_t e^{-\int_0^t (r_s + \lambda_s) ds} dt \right]$.
- Hence, the fair CDS spread can be written as $s_0 = \frac{\mathbb{E}_0 \left[ (1 - R) \int_0^T \lambda_t e^{-\int_0^t (r_s + \lambda_s) ds} dt \right]}{\mathbb{E}_0 \left[ \int_0^T e^{-\int_0^t (r_s + \lambda_s) ds} dt \right]}$ (weighted average default arrival).
- Under constant default arrival rates (flat CDS curve), we have $s_0 = \lambda (1 - R)$.
- Hence, annualized default probability (rate) $\lambda = s_0 / (1 - R)$.
- All expectations are under $\mathbb{Q}$. 
Interactions between interest rates and credit spreads

- If we only need to value the premium leg, we can model the dynamics of the defaultable rates $x_t = r_t + \lambda_t$ directly.
- To value the protection leg (and derive the CDS spread value), we need to separately model the dynamics of $r_t$ and $\lambda_t$.
- Technically, if we model both $r$ and $\lambda$ as affine functions of some affine factors $X_t$, we would have tractable exponential-affine solutions as described earlier: $r_t = a_r + b_r^\top X_t$, $\lambda_t = a_i + b_i^\top X_t$.
- Technically you can add any interactions you want, but practically they must be economically sensible and preferably parsimonious.
  - The benchmark rate $r$ is a market-wide variable, it is appropriate to start with $r$ and specify how each firm’s credit depends on the market:
    \[
    \begin{align*}
    r_t &= a_r + b_r^\top X_t, \\
    \lambda_t &= a_i + b_i^\top X_t + c_i^\top Y_t \\
    dX_t &= (\theta_x - \kappa_x X_t) dt + dW_{xt}, \\
    dY_t &= (\theta_y - \kappa_{xy} X_t - \kappa_y Y_t) dt + dW_{yt},
    \end{align*}
    \]
    $Y_t$ is firm-specific credit risk. Note the two types of interactions (contemporaneous v. predictive).
A more parsimonious model design

*Everything should be made as simple as possible, but not simpler.*

- A 3-factor benchmark rate dynamics:
  
  \[
  dr_t = \kappa_r (m_t - r_t) \, dt + \sigma_r dW^1_t,
  \]
  
  \[
  dm_t = \kappa_r s_{\kappa} (l_t - m_t) \, dt + \sigma_r dW^2_t, \quad s_{\kappa} < 1,
  \]
  
  \[
  dl_t = \kappa_r s_{\kappa}^2 (\theta - l_t) \, dt + \sigma_r dW^3_t.
  \]

  where \( dW^i \) are independent of each other, with identical market price \( \gamma_r \).

- Dimension-invariant cascade structure (Calvet, Fisher, Wu): Add as many factors as fitting needs, without adding parameters.

- Near perfect fitting of the benchmark curve is important for pricing CDS.

- Relating firm-specific CDS spreads to the benchmark curve
  
  \[
  \lambda^i_t = \beta^\top_i X_t + y_t,
  \]
  
  \[
  dy_t = \kappa_y (c_t - y_t) \, dt + \sum_{j=1}^3 \kappa^{ij}_{yx} (x_{j+1,t} - x_{j,t}) \, dt + \sigma_y dW^4_t,
  \]
  
  \[
  dc_t = \kappa_c (\theta - c_t) \, dt + \sum_{j=1}^3 \kappa^{ij}_{cx} (x_{j+1,t} - x_{j,t}) \, dt + \sigma_y dW^5_t, \quad \kappa_c < \kappa_y.
  \]

- Both contemporaneous projection and predictive interaction in a parsimonious setting.
Interest rates can be different at different terms (maturities). The pattern across different terms is referred to as the *term structure behavior*.

Generically speaking, the interest rate term structure can be decomposed into three components:

2. **Risk premium**: Long-term bonds are more sensitive to rate movements (and riskier), and hence higher returns (yields) on average.
3. **Convexity**: The bond value dependence on rate is convex, more so for long-term bonds. High rate volatility benefits long bond holders. As such, they can afford to ask for a lower expected return (yield).

The decomposition can be most clearly seen from the forward rate relation under a simplified Vasick model:

\[
dr_t = \sigma dW_t, \quad f(t, \tau) = r_t - \gamma \sigma \tau - \frac{1}{2} \sigma^2 \tau^2
\]

The three-component term structure decomposition works whether the “short rate” \( r \) is default-free or default able.
Summary: Short-rate risk decomposition

- In many applications, it is useful to decompose the short rate into several components and model the dynamics of each component separately/jointly.
  1. General affine: Short rate is decomposed into a linear combination of several “factors” — Factors can rotate.
  2. Dimension-invariant cascade: Short rate dynamics is decomposed into different frequency components.
  3. Credit risk: The short rate for defaultable bond can be decomposed into a benchmark component and a credit spread.
     - The traditional benchmark tends to be a “default-free” rate, now most studies switch to a practical benchmark actually used in the industry, which is not necessarily default free.
  4. Liquidity risk: Liquidity risk can also be separated out as a short rate spread against some more liquid benchmark.
  5. Benchmark rate, credit spread, liquidity spread can each be further decomposed, for example, into different frequency components.

- The exact decomposition depends on the type of questions one wants to address, always with identification in mind.
- The technicality (tractability) behind the bond pricing is the same, regardless of the economic meanings of the decomposition.
Outline

1. The term structure
   - Introduction
   - Dynamic term structure models
   - The impact of defaults on the term structure
   - Pricing CDS as an example
   - Interactions between interest rates and credit spreads

2. The capital structure
   - Interactions between capital structure decisions and default probability
   - The Merton (1974) model
   - Beyond Merton: Limitations and extensions
   - Default effects on equity option pricing
   - Two types of defaults with different market responses and impacts
   - Disentangling the different effects
In the previous sections, we specify interest and credit risk dynamics and derive interest rate and credit spread term structure. But what dictates the credit risk (and its dynamics) of a firm? The starting point must be the company’s capital structure: If a company does not have any liability, it has nothing to default on. The larger the liability (the higher the leverage), the more likely the firm can have difficulties meeting its liability obligations in the future.

Different types of business operations can accommodate different levels of leverage, for the same degree of risk.

- **Business risk:** If you know how much money you need and you can generate at each point in time, you can borrow up to that point, with no chance of default. Otherwise, you need to leave cushions.
- **Refinancing flexibility/risk:** If you can borrow new debt (or raise equity) to pay old debt, you don’t need to default, either.
- **Investment flexibility:** If you can freely adjust your investment position (asset characteristics), the chance of default is much smaller.

Capital structure is not (always) a static feature of a firm, but a decision/control variable that the managers can actively work on/adjust.
The Merton (1974) model: Set up

- Assumptions:
  - The company has a zero-coupon bond with principal $D$ and expiring at $T$. Company defaults if and only if its asset value at time $T$, $A_T$ is less than the debt principal.
  - Asset values follows a geometric Brownian motion (GBM): $dA_t/A_t = \mu dt + \sigma_A dW_t$.

- Comments:
  - The company does not have the flexibility of adjusting leverage, refinancing, or changing investments/business.
  - The model captures two key elements of default risk: leverage and business risk, in a very simple and intuitive way.
  - Models that start with the capital structure description and asset value dynamics are often referred to as “structural models,” in contrast with “reduced-form” models that directly specify default arrival rate dynamics.
  - Don’t trust any arguments that favor one against the other unconditionally. Each type of model serves a different purpose, and the distinction/definition is not as clear cut as you think.
The Merton (1974) model: Pricing

- Under Merton (1974), equity is a European call option on the asset: At debt maturity $T$, the equity holders receive $\max(0, A_T - D)$.

- Under the GBM dynamics, the call-option (equity) value satisfies the Black-Merton-Scholes formula: $E_t = A_t N(d + \sigma_A \sqrt{T-t}) - DN(d)$, where $d$ is a standardized variable that measures the number of standard deviations by which the log debt principal is below the conditional risk-neutral mean of the log asset value,

\[
d = \frac{\mathbb{E}_t(\ln A_T) - \ln(D)}{\text{Std}_t(\ln A_T)} = \frac{\ln(A_t) + r(T-t) - \frac{1}{2} \sigma_A^2 (T-t) - \ln(D)}{\sigma_A \sqrt{T-t}}.
\]

- In the option pricing literature, $d$ is referred to as the standardized moneyness of the option.

- In the credit literature, $d$ is often referred to as the distance-to-default, a critical input for both Moody’s and RMI’s credit model.

- The numerator measures the expected financial leverage of the firm at the debt expiration, and the denominator measures the uncertainty (standard deviation) of this leverage — Distance to default is essentially a standardized financial leverage measure that is comparable across firms with different business risks.
The key contribution of the model is the distance to default measure, which standardizes the capital structure (financial leverage) to make it comparable across firms with different business risk profiles.

Many academic studies debate on whether the credit spread implied by the model matches market values, and call the mismatch a “puzzle.”

Moody’s and RMI are more practical and use the distance to default to differentiate the credit qualities of different firms:

- One can always regress credit spread on leverage and risk, but Merton provides an intuitive, structural way of combining these two firm characteristics into one standardized variable that becomes much more comparable across firms than each variable alone.

Homework: Compare out-of-sample performance:

- Each day, split firms into two halves. Estimate a bivariate regression of credit spreads against financial leverage and volatility on one sample and check its out-of-sample performance.
- Perform a univariate regression of credit spreads on distance to default and check its out-of-sample performance.
Beyond Merton: Limitations and extensions

- The key characteristics of a great model is that it makes bold/dramatic simplifying assumptions to arrive at deep insights that make economic sense.

- One key objective of “structural” credit modeling is to see through the millions of details in a firm and show how different firm characteristics combine to determine credit risk.

  - The Merton model says that the key credit determinants are financial leverage and business risk and their contributions do not come in additively, but one serves as a scaler of the other.

- Given the highly stylized nature, the model can of course be extended in several dimensions:
  1. Asset value dynamics
  2. Capital structure specifications
  3. Default triggering mechanisms
  4. Firm operation, investment, and refinancing decisions
Everybody talks about jumps and stochastic volatility. The question is not the process itself, but how these features show up in firm characteristics and in the transformation.

- Jumps and stochastic volatility changes the return distribution. $1 - N(d)$ captures the default probability under Merton. The normal transformation can be switched to a non-normal transformation, but such variations only generate marginal effects: The transformation won’t change the cross-sectional ranking of credit risk across firms.

- The effects of sudden drops (jumps) in asset value can be translated into sudden drops in stock price, which can show up vividly in deep-out-of-money (DOOM) put options (Carr, Wu, 2011 RFS).

- The fact that volatility varies over time suggests that one should use short rolling window in estimating stock return volatilities or use option-implied volatility as a forward-looking measure that fully captures the time variation.
Beyond Merton: Capital structure specification

Virtually all firms have debt of multiple maturities, including coupon/interest expense payments in the middle.

- Even if one uses the Merton model, how to implement the model remains an issue. For example, KMV uses short-term liability + half of long-term liability to proxy the debt principal $D$ in Merton model. Why?

- Given the same amount of total debt, which types of firms have higher credit risk at different horizons? Firms with more short-term or more long-term debt?

- These questions cannot be appropriately addressed theoretically without making the right assumption on the default triggering mechanism, which may also depend on the (flexibility) of the refinancing decision.
  
  - Do firms only default at debt payment period or can they default at any time? Can debt holders force bankruptcy preemptively?
  - When one debt matures, should/will the firm pay off the debt with its current earnings (by paying less dividend), its asset (via liquidation), or refinancing (via equity or debt)?
Beyond Merton: Debt payment decisions

- If the firm plans to payoff its obligation using earnings, it better has enough earnings to cover the payment — *Interest coverage ratio* relies on this idea.

- If the firm plans to liquidate asset to cover its debt payment, its asset must have a significant component that can be liquidated easily — *liquidity ratios* are useful indicators.
  - Investment firms can reduce their investment size fairly easily and they indeed do so frequently according to market conditions; but it can be costly for a manufacturing firm to do so.

- If the firm plans to refinance its expiring debt, market credit conditions can become an important concern. Also, all variables that are considered by debt investors (profitability, existing leverage, liquidity, size, past performance, etc) become naturally important in determining the refinancing cost (and possibility).

- There are many structural models that go beyond Merton, such as Leland, Geske, etc. You can also try to think of new models. Practically, the key is whether these extensions allow you to incorporate more useful information observed on the firm.
Triggering mechanism: A barrier option approach

- Merton assumes that default can only be triggered at the debt expiry.
- An alternative is to assume that the firm can default any time before the debt maturity when the firm’s asset value falls below a certain threshold $B$.
  $\Rightarrow$ Equity becomes a call option on the asset value with a knock-out barrier.
  - The generic down-and-out call option formula is a bit complicated. Let $C(S, K, T)$ denote the vanilla call option value at spot $S$, strike $K$, and expiry $T$, the down-and-out call value is (assuming $B \leq D$),
    $$DOC_t(A, D, B, T) = C_t(A, D, T) - \left(\frac{A}{B}\right)^{2\alpha} C\left(\frac{B^2}{A}, D, T\right), \quad \alpha = \frac{1}{2} - \frac{r}{\sigma^2}.$$  
  - Assume zero rates ($r = 0$) and set the barrier to the debt principal $B = D$, the down-and-out call option (the equity value) is always worth its intrinsic,
    $$DOC_t(A, D, D, T) = \max(0, A - D)$$
- Bharath and Shumway (2008)’s “naive” Merton alternative assumes $E = A - D$ and can be justified under this barrier assumption.
- Leland (1994) and Leland and Toft (1996) consider more complex setups with barriers. KMV claims to use/consider a barrier approach.
Multiple debts: A two-debt example

- Merton assumes that the firm has one zero-coupon bond expiring at $T$.
- Now let’s consider the case of two zero-coupon bonds of principals $D_1$ and $D_2$ expiring at $T_1$ and $T_2$, respectively ($T_1 < T_2$).
  - At $T_2$, the default condition is the same as in Merton: The firm defaults if at that time $A_{T_2} < D_2$, and equity is the residual claimer.
  - At $T_1$, the default condition is often assumed to be $A_{T_1} < D_1 + E_{T_1}(D_2)$.
  - Although the firm only needs to pay $D_1$ at $T_1$, it is unlikely the firm can roll-over its debt and maintain its firm size and debt structure if the firm’s value is less than its debt value.
- The equity can be regarded as a compound option (Geske, 1974, 1977).
- The two-debt example can be solved semi-analytically, but it is probably more practical to build a binomial tree, with which one can match the actual debt payment schedules and consider additional conditions.
  - In practice, the conditions can be stricter, say a fraction ($\delta$) of its asset value must be higher than the debt value.
  - More research is needed on how to model the firm-level liquidity effect (how easy it is to liquidate its asset) and market-level credit crunch (how easy it is to obtain refinancing).
Free of default via flexible leverage rebalancing

- Merton and Geske assume that default only occur at debt payment times.
  - These companies have a passive debt structure that will only be updated upon debt expiry.
- Financial firms, including banks and investment firms, can be much more active in rebalancing their financial leverage (Adrian and Shin (2010)).
- Think of the simplest example of trading on margin.
  - Trading on margin is essentially levered investment. The asset is the amount of the investment, which can be several times higher than the equity (the amount of margin one puts at the exchange).
  - One rarely observes default on margin trading, despite high leverage.
  - The key is that it has a barrier feature at which point the creditor (exchange) can force close the position if no new capital is injected. The barrier is set at a level that the creditor rarely loses money at the forced closure.
  - The debt can be thought of as short-term credit.
- If asset follows diffusion dynamics and the firm can flexibly (and optimally) adjust the financial leverage, the firm in principal never needs to default. Investment flexibility can play a similar role.
Constant proportional portfolio insurance (CPPI)

This may be a side topic, but it is somewhat useful in terms of how we think of credit risk, especially for financial firms.

- CPPI allows an investor to limit downside risk while retaining some upside potential by maintaining an exposure to risky assets equal to a constant multiple $m > 1$ of the cushion. In diffusion models with continuous trading, this strategy has no downside risk.
- CPPI is a self-financing strategy, with the goal to guarantee a fixed amount $N$ of capital at maturity $T$.
- Let $B_t$ denote the present value of the guaranteed amount $N$ and $V_t$ denote the total portfolio value. The strategy at any date $t$ can be described as,
  - If $V_t > B_t$, the risky asset exposure (amount of money invested into the risky asset) is given by $mC_t = m(V_t - B_t)$, where $C_t$ is the “cushion” and $m > 1$ is a constant multiplier.
  - If $V_t \leq B_t$, the entire portfolio is invested into the zero-coupon.
- If one can guarantee a fixed amount at expiration, one never needs to default.
- Cont and Tankov (2007) investigate the downside risk of a CPPI strategy when there are downside jumps.
Under the Merton model, the diffusion nature of the dynamics dictates that the default event is completely predictable before it happens.

We show via the CPPI strategy that one can always avoid the predictable event if one is allowed to scale the risky investment freely.

Hence, the true risk under the Merton model is not from the diffusion dynamics, but from structural constraints — the firm cannot do anything to get out once in the debt.

However, when the asset value can jump down unexpectedly by a large amount, even frequent rebalancing cannot guarantee default free.

Actions no longer help, it is fate.

Thus, in practice, one should be worried about large downside jumps, while working actively to avoid difficulties caused by diffusion movements (via active rebalancing).
A new structural model
Equity as a cancelable American call option

- The debt has a principal $D$, expiry $T$, and a continuously paid coupon.
- The coupon rate is equal to the contemporaneous risk-free rate $r_t \geq 0$, which evolves randomly over time. Hence, the company has a floating rate debt with a continuous coupon of size $r_t D$ dollars per unit time.
- Equity holders receive dividends continuously, with $q_t$ as the dividend yield.
- Prior to default, the coupons to the debt holders and the dividends to the shareholders are financed by any combination of asset sales or additional equity issuance.
- At any time, the equity holders can choose to stop paying the coupon by declaring bankruptcy. This declaration eliminates all future dividends, causing the share price to hit zero.
- Equity holders can be viewed as holding a “cancelable” American call option on the firm value with strike $D$ and maturity $T$. 
A structural model with a fatal default corridor

- Like a standard American call, the cancelable call has the value \((A_T - D)^+\) at expiry. Unlike a standard American call, the cancelable call has a holding cost, given by \(r_tD - q_tA\) per unit time.

- The call is **cancelable** because the shareholders can at any time cancel it by refusing to pay any more coupons, at the cost of foregoing subsequent dividends and the optionality.

- The cancelable call (equity) value can be linked to the standard American put value at the same strike and maturity via the following parity condition:

  \[ C_t^C(D, T) = A_t - D + P_t(D, T). \]

- The floating debt value is \(F_t(D, T) = A_t - C^t(D, T) = D - P_t(D, T)\).

- Defaultable displaced dynamics: Assume the firm value will stay above \(B_1\) before default and drop below \(B_2\) after default, with \(B_2 < D < B_1\). Then it is never optimal to exercise the American put option on the firm before the firm defaults and it is optimal to exercise upon default.

- This leads to an analogous **default corridor** on the equity value at \([0, B_1 - D]\).

- Default is purely triggered by outside shocks.
Default effects on equity option pricing

- The Merton (1974) structural model is very insightful on how financial leverage and business risk combine to contribute to default risk.

- To analyze how default affects equity option pricing, Merton switches to a different framework in his 1976 paper by directly assuming that the equity value follows a geometric Brownian motion prior to default and drops to zero upon default.

- He further assumes that default arrives unexpectedly via a Poisson process with arrival rate $\lambda$.

- These assumptions are both convenient (tractable) and reasonable, and serve a completely different purpose from the 1974 model. Both are classic.

  - If you want to model the impact of A on B, you start with A dynamics. If you want to model the impact of B on A, you start with B dynamics. Neither is more fundamental or reduced than the other.

  - Henceforth, I classify “structural models” as models that describes the capital structure. If you do not want to discuss capital structure as your question, you do not need to model capital structure.
The Merton (1976) jump to default model

- Conditional on no default, the stock price follows the geometric Brownian motion just as in the Black-Merton-Scholes (BMS) model.
- Upon default, equity value drops to zero, so does call option value.
- Call option pricing remains extremely tractable: One can still use the Nobel-winning BMS formula, except that one needs to replace the risk free rate $r$ with $r + \lambda$.
- An approximation of the implied volatility shows the default effect:
  \[ IV_t(d, T) \approx \sigma + \frac{N(d)}{N'(d)} \sqrt{T - t} \lambda, \quad d = \frac{\ln F_t/K - \frac{1}{2} \sigma^2 (T-t)}{\sigma \sqrt{T-t}}. \]
  - For firms with default risk, the “volatility risk premium” ($IV - \sigma$) is not necessarily a measure of risk premium for volatility, but more likely a measure of default risk — A few papers document the predictability, but with the wrong interpretation.
  - The probability of default generates a negatively skewed implied volatility smile: Low strike options have higher implied volatilities due to their higher exposure to default risk.
  - Another measure of the default probability: The implied volatility smile slope at $d = 0$. 
Rubinstein (1983) displaced diffusion model

A useful distraction:

- The Merton jump to default (MJD) model can generate a negatively sloped implied volatility smirk, but not a smile.

- Rubinstein (1983) introduces the displaced diffusion (RDD) model,

\[ S_t = Be^{rt} + (S_0 - B)e^{rt + \sigma W_t - \frac{1}{2} \sigma^2 t}, B > 0 \]

One can think of the firm as putting some asset in cash \((B)\) to earn risk-free rate while investing the rest in a risky asset that follows GBM.

- The cash investment suggests that the stock value will never go below \(Be^{rt}\), and hence has a non-zero lower bound.

- Option pricing still takes the form of the BMS formula, with some adjustment, \(C(S, K) = BMS(S - B, K - B)\).

- This shifting (displacement) generates an upward sloping (positively sloped) implied volatility smile,

\[ IV(K) \approx \frac{\ln(K/S)}{\ln((K-B)/(S-B))} \sigma, \quad K > B. \]
Carr-Wu defautable displaced diffusion (DDD) model

Combining Merton’s jump to default idea with Rubinstein’s displaced diffusion, Carr and Wu (2011, RFS) propose a default able displaced diffusion model.

- Prior to default, the stock price follows RDD and hence stays above $B$.
- Upon default, the stock price drops 0 (or some equity recovery value $R$).
- There is a default corridor $[0, B]$ that the stock price can never reside in.
- This model can be regarded either as reduced-form or structural (recall our earlier structural model with floating debt).
- In the presence of the default corridor, one can use a deep-out-of-money (DOOM) American put stock option struck within the corridor to create a pure credit contract that has nothing to do with delta, vega...

  - $URC = P(K, T)/K$ (with $K \leq B$) denotes a unit recovery claim that pays one dollar when and only when the firm defaults!
  - If the corridor has a non-zero lower bound (non-zero equity recovery), we can choose two put options struck within the corridor to create the unit recovery claim, $URC = (P(K_2) - P(K_1))/(K_2 - K_1)$.
  - The URC implications hold, regardless of pre-default or post-default dynamics... It only depends on the presence of a default corridor.
Fundamentally, there are two types of defaults that ask for different types of responses and have different levels of market impacts.

1. **Structural defaults**: Defaults due to structural constraints.
   - Both creditors and debtors can see default coming, but they are structurally constrained to do anything about it.
   - The Merton (1974) model is one such case: The diffusion behavior dictates that default event is predictable, yet creditors are not allowed to force close the debt early to avoid loss of principal.
   - The barrier option alternative assumption allows the creditor to get out early as soon as the asset value hits a boundary. Under certain assumptions, the creditor never loses its principal.
   - The CPPI example is inspirational in the sense that the financial managers can drastically reduce their chance of default (to zero) under diffusion dynamics if they can actively rebalance their portfolio/leverage — All firms have some capacity to do so, more for financial firms (such as investment banks) and investment firms.

To avoid defaults caused by structural constraints, financial managers should strive to keep credit channels open, maintain a flexible capital structure that can be readily updated/adjusted based on market conditions.
Two types of defaults

Fundamentally, there are two types of defaults that ask for different types of responses and have different levels of market impacts.

1. **Defaults induced by structural constraints**

2. **Default induced by exogenous sudden, large, market shocks**:

   - It is difficult to neutralize such shocks via dynamic hedging, which works better for diffusions or jumps of fixed/known sizes than for jumps of random sizes.
   - One can in principle take (many) option positions to hedge jumps of different sizes — The strong aggregate demand for such insurance-like option contracts often pushes the option price to very high levels.
   - Worse yet, large negative shocks often generate chain reactions, or “self-exciting behavior:” One large negative shock tends to increase the chance of having more large negative shocks to follow, either sequentially for one firm or one market (destabilizing spiral), or cross-sectionally across different firms or markets (contagion).

*World-wide crash-o-phobia* (Foresi & Wu:2005): Deep out-of-money put options are very expensive on market indexes across the world, more so for longer-term options, in defiance of central limit theorem.
The current status of equity option pricing

- The current status of the equity option pricing literature: a model with both jumps and stochastic volatility, e.g. Bates (1996), Baskhi, Cao, Chen (1997)

\[
\frac{dS_t}{S_t} = \sqrt{v_t} dW_t + \int_{\mathbb{R}_0} (e^x - 1)(\mu(dx, dt) - \pi(x)dx dt),
\]

\[
dv_t = \kappa(\theta - v_t) dt + \omega \sqrt{v_t} dZ_t, \quad \rho dt = \mathbb{E}[dW_t dZ_t] < 0
\]

- There can be one or two volatility factors; jump arrival rates can be constant or stochastic (Bates (1996, 2000), Pan (2000), Huang and Wu (2004)).

- The return-variance correlation $\rho$ is often assumed negative and referred to as the “leverage effect” (Black (76)):
A firm with a fixed amount of debt will have a higher financial leverage when the stock price goes down. The increased financial leverage increases the equity risk and hence its volatility.

- The focus is to capture the implied volatility skew.

- Puzzles:

  - Even all-equity firms seem to have a “leverage effect” (negative return-vol correlation) — The “leverage effect” is obviously not from the firm’s financial leverage if the firm does not have debt.

  - “Leverage effect” (negative implied volatility skew) seems to be stronger for market stock indexes than for single-name stocks.
The leverage effect may not come from financial leverage

- Fundamentally, the negative correlation between return and volatility may not necessarily come from the so-called leverage effect.

- *Volatility feedback effect* can also generate negative return-variance correlation. When a firm’s business risk receives a positive shock, the discount rate becomes larger, the valuation becomes lower, hence generating a negative relation between risk and valuation.
  - The effect is on the firm value level, regardless of financial leverage.
  - The feedback effect can be stronger for market indexes than for individual stocks because increases in idiosyncratic risk may not increase the discount rate.

- The *self-excitng market crashes* can generate a strong negative return-volatility relation as a destabilizing spiral.
  - The jump generates a negative skew in the distribution instantaneously; the self-exciting behavior further induces another layer of negative skewness at longer horizons via the jump-intensity interaction.

- *Financial leverage variation* does not necessarily lead to a negative return-variance correlation — Black (1976) treats the financial leverage variation as a passive effect of a market shock, but financial managers can also alter the leverage *proactively* to counter the impact of market shocks.
Disentangling the different effects via options

There is a chance that we can design an equity option pricing model accounting for these different effects separately and disentangle them via estimation using equity options across multiple strikes, maturities, and calendar times.

- To capture **structural defaults**, we must model how financial leverage varies with market conditions — The more proactively a firm can do this, the less likely it will default structurally.

- To model **market crashes**, one must account for the self-exiting effect — One negative shock can be bad, but one after another is debilitating.

- To disentangle business risk variation (and **volatility feedback effect**) from the financial leverage effect, one must separate the asset value dynamics from the financial leverage variation.

Such a model necessarily contains capital structure features (and hence structural), but it must also be reduced-form enough to price tractably securities along the whole capital structure (equity, debt, and their derivatives).

- Each day, one observes one stock price, but hundreds of stock options across different strikes and maturities. Incorporating the options information can enhance the separate identification of different mechanisms.

- Credit contracts (such as CDS term structure) and firm fundamental information (such as capital structure) can also be useful.
The model: Separate the financial leverage variation from the asset value dynamics

- Decompose the forward value of the equity index $F_t$ into a product of the asset value $A_t$ and the equity-to-asset ratio (EAR) $X_t$,

  $$ F_t = A_t X_t. \quad \Leftarrow \text{This is just a tautology,} \quad (3) $$

- Model leverage $X_t$ as a stand-alone CEV process:

  $$ \frac{dX_t}{X_t} = \delta X_t^{-p} dW_t, \quad p > 0. \quad (4) $$

  - **Leverage effect**: A decline in $X$ increases leverage [by definition], reduces equity value [via (3)], and raises equity volatility [via $X^{-p}$].

  - **Structural link**: When $A$ is fixed, the equity return volatility becomes a pure power function of the leverage ratio. A similar (more restricted) volatility structure can be obtained from the structural model of Leland (94).

  - **Firms can (and do) actively manage their leverages (Adrian & Shin (2008)).**
Model the asset value $A_t$ separately, in addition to leverage variation,

$$
\frac{dA_t}{A_t} = \sqrt{\nu_t^Z} dZ_t + \int_0^\infty (e^x - 1) \left( \mu^+ (dx, dt) - \pi_j^+ (x) dx \nu_t^J dt \right) + \int_{-\infty}^0 (e^x - 1) \left( \mu^- (dx, dt) - \pi_j^- (x) dx \nu_t^J dt \right),
$$

$$
d\nu_t^Z = \kappa_Z \left( \theta_Z - \nu_t^Z \right) dt + \sigma_Z \sqrt{\nu_t} dZ_t^\nu, \quad \mathbb{E} [dZ_t^\nu dZ_t] = \rho dt,
$$

$$
d\nu_t^J = \kappa_J \left( \theta_j - \nu_t^J \right) dt - \sigma_J \int_{-\infty}^0 x \left( \mu^- (dx, dt) - \pi_j^- (x) dx \nu_t^J dt \right).
$$

- **Volatility feedback** — $\rho < 0$.
- **Self-exciting crashes** — $\sigma_J > 0$. Negative jumps in asset return are associated with positive jumps in the jump arrival rate $\nu_t^J$.
  Jump specification: Variance gamma high-frequency jump:
  $$
  \pi^+_j (x) = e^{-x/\nu^+_j} x^{-1}, \quad \pi^-_j (x) = e^{-|x|/\nu^-_j} |x|^{-1}.
  $$
- When $X$ is fixed, the equity dynamics follow the asset dynamics.
  - $\rho < 0$ is often referred to as the leverage effect.
The stock price dynamics:

- The stock price dynamics:
  \[
  \frac{dF_t}{F_t} = \delta \left( \frac{F_t}{A_t} \right)^{-p} dW_t + \sqrt{\nu_t^Z} dZ_t + \int (e^x - 1) (\mu(dx, dt) - \pi(x)dxv_t^J dt).
  \]

- Linking/extending 3 strands of literature:
  1. The local volatility effect of Dupire (94):
     - Scaling \( F_t \) by \( A_t \) (both in dollars) makes the return variance a unitless quantity, and renders the dynamics scale free.
     - Power dependence on leverage is supported by Leland (94).
  2. The stochastic volatility of Heston (93):
     - Used purely for volatility feedback, regardless of leverage.
  3. The high-frequency jump of Madan, Carr, Chang (98):
     - Arrival rate varies stochastically, and jumps synchronously with VG jump in return.
Let $\nu^X_t = \delta^2 X_t^{-2p}$. The stock price dynamics can be written as a three-factor stochastic volatility model:

$$dF_t/F_t = \sqrt{\nu^X_t} dW_t + \sqrt{\nu^Z_t} dZ_t + \int_{\mathbb{R}_0} (e^x - 1) \left( \mu(dx, dt) - \pi(x) dx \nu^J_t dt \right).$$

where

$$d\nu^X_t = \kappa_X (\nu^X_t)^2 dt - \sigma_X (\nu^X_t)^{3/2} dW_t,$$

$$\leftarrow \text{a 3/2-process.} \quad (5)$$

with $\kappa_X = p(2p + 1)$ and $\sigma_X = 2p$. Henceforth, normalize $\delta = 1$.

- **Financial leverage variation is a separate source of stochastic volatility for stock return.**

- The 3/2-vol of vol dependence in (5) has been shown to perform better than square-root dependence, e.g., Bakshi, Ju, Yang (2006).

- The model can be represented either as a local vol model with level dependence or a pure scale-free stochastic volatility model without level dependence — unifying the two strands of literature.
Active managerial decisions on financial leverage

- The specifications so far are on the $\mathbb{Q}$ dynamics (Recall my $\mathbb{Q}$ to $\mathbb{P}$ suggestion)
  - The forward price, forward asset value, and leverage ratio $X_t$ are all assumed to be martingales under $\mathbb{Q}$.
- Their deviations from the $\mathbb{P}$ dynamics reflect the market prices of risks.
- **Simplifying assumptions**: Constant market prices ($\gamma^\nu, \gamma^J$) for diffusion variance risk ($Z_t$) and jump risk ($J_t$).
- **Structural decisions**: Financial managers make financial leverage decisions based on the current levels of all three types of risks:
  \[
dX_t = X_t^{1-p} \left( aX_t - \kappa_{XX}X_t - \kappa_{XZ}v_t^Z - \kappa_{XJ}v_t^J \right) dt + X_t^{1-p} dW_t^\mathbb{P}.
\]
  Market price of $W_t$ risk is $\gamma^X_t = aX_t - \kappa_{XX}X_t - \kappa_{XZ}v_t^Z - \kappa_{XJ}v_t^J$.
  - $\kappa_{XX}$: Mean reversion, leverage level targeting.
  - $\kappa_{XZ}$: Response to diffusion business risk.
  - $\kappa_{XJ}$: Response to jump business risk.
- **Empirical findings**: Firms increase financial leverage when diffusion risk increases, but reduce leverage when jump risk increases — They can dynamically hedge diffusion risk, but are really worried about crash risk.
Summary

- Modeling default-free and default-able term structures is virtually the same.
  - The decomposition is artificial and is now mostly based on a decomposition of benchmark (instead of riskfree) rate plus spreads due to credit and liquidity...

- Defaults can come from either structural constraints or large unexpected negative market shocks.
  - Firms with more financial flexibility (less structural constraints) can drastically reduce their probability of structural default via dynamic rebalancing of their capital structures based on market conditions.
  - Large negative unexpected market shocks are not only difficult to hedge and expensive to insure against, but also often generate self-exciting spirals both sequentially and cross-sectionally.

- Disentangling the different effects require that
  - Modelers come out of the traditional mindset (of structural v. reduced-form dichotomy) and organically combine the benefits of both.
  - Estimation includes financial securities along the whole capital structure, as well as their derivatives.
Ultimately, one wants to link together the following three pieces:

1. Firm fundamental characteristics such as leverage, profitability, liquidity, business risk...
2. Security prices along the capital structure and term structure
3. Default probability predictions and capital/term structure valuations

How one slices/dices these pieces depends on a number of factors:

- **Coverage**: Information that is available only for a small number of firms cannot be directly used as predictors for universal valuation/prediction. They can still serve as benchmarks in the relation construction process.
- **Quality**: An item that is very useful in theory can be totally useless if the observation contains lots of noise.
  - Some researchers’ desire for avoiding market prices may originate from their concerns that market prices can break down during crucial/crisis times. This is more so for less liquid, more exotic derivative contracts.
  - Exotic derivative contracts also have a clientele issue: Not everyone can trade these contracts. Their prices reflect the valuation of the traders, but may not reflect the view of the general market.

Use as much information as possible, while try to find methodologies to get around the above concerns.