Modeling Financial Security Returns with Time-Changed Lévy Processes

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Courant Institute Math Finance Seminar, February 26, 2008
Objectives

- How to model financial security returns using time-changed Lévy processes
  - with an eye on data.

- How to price options based on these models
  - with an eye on numerical efficiency.

- How to estimate these models
  - with an eye on different applications:
    - market-making,
    - long-term convergence trading,
    - risk-premium taking for systematic risk exposure,
    - academics.
Why time-changed Lévy processes?

Key advantages:

- **Generality:**
  - Lévy processes can generate almost any return innovation distribution.
  - Applying stochastic time changes randomizes the innovation distribution over time \(\Rightarrow\) stochastic volatility, correlation, skewness, ....

- **Explicit economic mapping:**
  - Each Lévy component \(\leftrightarrow\) shocks from one economic source.
  - Time change captures the time-varying intensity of its impact.
  \(\Rightarrow\) makes model design more intuitive, parsimonious, and sensible.

- **Tractability:** A model is tractable for option pricing if we have
  - tractable characteristic exponent for the Lévy components.
  - tractable Laplace transform for the time change.
  \(\Rightarrow\) Any combinations of the two generate tractable return dynamics.
A Lévy process is a continuous-time process that generates stationary, independent increments ...

Think of return innovation in discrete time: $R_{t+1} = \mu_t + \sigma_t \varepsilon_{t+1}$.

Lévy processes generate iid return innovation distributions via the Lévy triplet $(\mu, \sigma, \pi(x))$. ($\pi(x)$–Lévy density).

The Lévy-Khintchine Theorem:

$$
\phi_{X_t}(u) \equiv \mathbb{E}[e^{iuX_t}] = e^{-t\psi(u)}, \quad u \in \mathcal{D} \subseteq \mathbb{C}
$$

$$
\psi(u) = -iu\mu + \frac{1}{2}u^2\sigma^2 + \int_{0}^{\infty} (1 - e^{iux} + iux1_{|x|<1}) \pi(x)dx,
$$

Innovation distribution $\leftrightarrow$ characteristic exponent $\psi(u) \leftrightarrow$ Lévy triplet

- Constraint: $\int_0^1 x^2 \pi(x) dx < \infty$ (finite quadratic variation).
- Tractable: The integral can be carried out explicitly.
Tractable examples

1. Brownian motion (BSM) \((\mu t + \sigma W_t)\): normal shocks.

2. Compound Poisson jumps (Merton, 76): Large but rare events.

\[
\pi(x) = \lambda \frac{1}{\sqrt{2\pi v_J}} \exp\left(-\frac{(x - \mu_J)^2}{2v_J}\right).
\]

3. Dampened power law (DPL):

\[
\pi(x) = \begin{cases} 
\lambda \exp(-\beta_+ x) x^{-\alpha-1}, & x > 0, \\
\lambda \exp(-\beta_- |x|) |x|^{-\alpha-1}, & x < 0,
\end{cases}
\]

- Finite activity when \(\alpha < 0\): \(\int_{\mathbb{R}_0^+} \pi(x) dx < \infty\). Compound Poisson. Large and rare events.
- Infinite activity when \(\alpha \geq 0\): Both small and large jumps.
- Infinite variation when \(\alpha \geq 1\): many small jumps,

\[
\int_{\mathbb{R}_0^+} (|x| \wedge 1) \pi(x) dx = \infty.
\]

\(\alpha \leq 2\) to guarantee finite quadratic variation.

Market movements of all magnitudes, from small movements to market crashes.
Analytical characteristic exponents

- Diffusion: \( \psi(u) = -i u \mu + \frac{1}{2} u^2 \sigma^2. \)

- Merton's compound Poisson jumps:
  \[
  \psi(u) = \lambda \left( 1 - e^{iu \mu - \frac{1}{2} u^2 \nu} \right).
  \]

- Dampened power law: ( for \( \alpha \neq 0, 1 \))
  \[
  \psi(u) = -\lambda \Gamma(-\alpha) \left( (\beta_+ - iu)^\alpha - \beta_+^\alpha + (\beta_- + iu)^\alpha - \beta_-^\alpha \right) - iu C(h)
  \]
  - When \( \alpha \to 2 \), smooth transition to diffusion (quadratic function of \( u \)).
  - When \( \alpha = 0 \) (Variance-gamma by Madan et al):
    \[
    \psi(u) = \lambda \ln \left( 1 - \frac{iu}{\beta_+} \right) \left( 1 + \frac{iu}{\beta_-} \right) = \lambda \left( \ln(\beta_+ - iu) - \ln \beta + \ln(\beta_- + iu) - \ln \beta_- \right).
    \]
  - When \( \alpha = 1 \) (exponentially dampened Cauchy, Wu 2006):
    \[
    \psi(u) = -\lambda \left( (\beta_+ - iu) \ln (\beta_+ - iu) / \beta_+ + \lambda (\beta_- + iu) \ln (\beta_- + iu) / \beta_- \right) - iu C(h).
    \]
  - \( \beta_\pm = 0 \) (no dampening): \( \alpha \)-stable law
Other Lévy examples

- Other examples:
  - The normal inverse Gaussian (NIG) process of Barndorff-Nielsen (1998)
  - The generalized hyperbolic process (Eberlein, Keller, Prause (1998))
  - The Meixner process (Schoutens (2003))
  - ...

- Bottom line:
  - All tractable in terms of analytical characteristic exponents $\psi(u)$.
  - We can use FFT to generate the density function of the innovation (for model estimation).
  - We can also use FFT to compute option values ...

- Question: Do we need Lévy jumps to model financial security returns?
  - It is important to look at the data...
Implied volatility smiles & skews on a stock

![Graph showing implied volatility smiles and skews on a stock. The x-axis represents moneyness, calculated as $\ln(K/F) / \sigma \sqrt{\tau}$, where $K$ is the strike price, $F$ is the price of the stock, $\sigma$ is the volatility, and $\tau$ is the time to maturity. The y-axis represents implied volatility. The graph shows short-term smile and long-term skew for different maturities: 32, 95, 186, 368, 732.](image-url)
Implied volatility skews on a stock index (SPX)

More skews than smiles

Maturities: 32  60  151  242  333  704

Moneyness = ln(K/F) / (σ√τ)

Implied Volatility

SPX: 17–Jan–2006
Average implied volatility smiles on currencies

Maturities: 1m (solid), 3m (dashed), 1y (dash-dotted)
The role of jumps at very short maturities

- Implied volatility smiles (skews) ↔ non-normality (asymmetry) for the risk-neutral return distribution (Backus, Foresi, Wu (97)):

\[ IV(d) \approx ATMV \left( 1 + \frac{\text{Skew.}}{6} d + \frac{\text{Kurt.}}{24} d^2 \right), \quad d = \frac{\ln K/F}{\sigma \sqrt{\tau}} \]

- Two mechanisms to generate return non-normality:
  - Use Lévy jumps to generate non-normality for the innovation distribution.
  - Use stochastic volatility to generates non-normality through mixing over multiple periods.

- Over very short maturities (1 period), *only jumps contribute to return non-normalities.*
Time decay of short-term OTM options


- As option maturity $\downarrow$ zero, OTM option value $\downarrow$ zero.
- The speed of decay is exponential $O(e^{-c/T})$ under pure diffusion, but linear $O(T)$ in the presence of jumps.
- Term decay plot: $\ln(OTM/T) \sim \ln(T)$ at fixed $K$:

$\text{In the presence of jumps, the Black-Scholes implied volatility for OTM options } IV(\tau, K) \text{ explodes as } \tau \downarrow 0.$
(II) The impacts of jumps at very long horizons

- Central limit theorem (CLT): Return distribution converges to normal with aggregation under certain conditions (finite return variance,...) ⇒ As option maturity increases, the smile should flatten.
- Evidence: The skew does not flatten, but steepens!
  - Return variance is infinite. ⇒ CLT does not apply.
  - Down jumps only. ⇒ Option has finite value.
- But CLT seems to hold fine statistically:
Reconcile $\mathbb{P}$ with $\mathbb{Q}$ via DPL jumps


- Model return innovations under $\mathbb{P}$ by DPL:

  \[
  \pi(x) = \begin{cases} 
  \lambda \exp(-\beta_+ x) x^{-\alpha - 1}, & x > 0, \\
  \lambda \exp(-\beta_- |x|) |x|^{-\alpha - 1}, & x < 0.
  \end{cases}
  \]

  All return moments are finite with $\beta_\pm > 0$. **CLT applies.**

- Market price of jump risk ($\gamma$): $\frac{dQ}{dP}\bigg|_t = \mathcal{E}(-\gamma X)$

- The return innovation process remains DPL under $\mathbb{Q}$:

  \[
  \pi(x) = \begin{cases} 
  \lambda \exp(-(\beta_+ + \gamma) x) x^{-\alpha - 1}, & x > 0, \\
  \lambda \exp(-(\beta_- - \gamma) |x|) |x|^{-\alpha - 1}, & x < 0.
  \end{cases}
  \]

- To break CLT under $\mathbb{Q}$, set $\gamma = \beta_-$ so that $\beta_-^Q = 0$.

- Reconciling $\mathbb{P}$ with $\mathbb{Q}$: **Investors pay maximum price on hedging against down jumps.**
When a company defaults, its stock value *jumps* to zero.

This default risk generates a steep skew in long-term stock options.

Evidence: Stock option implied volatility skews are correlated with credit default swap (CDS) spreads written on the same company.

Three Lévy jump components

I. Market risk (FMLS under $\mathbb{Q}$, DPL under $\mathbb{P}$)

II. Idiosyncratic risk (DPL under both $\mathbb{P}$ and $\mathbb{Q}$)

III. Default risk (Compound Poisson jumps).

- Stock options: Information and identification
  - Identify market risk from stock index options.
  - Identify the credit risk component from the CDS market.
  - Identify the idiosyncratic risk from the single-name stock options.

- Currency options:
  - Model currency return as the difference of two log pricing kernels (market risks).
  - Default risk also shows up in FX for low-rating economies.

Economic implications

- In the Black-Scholes world (one-factor diffusion):
  - The market is complete with a bond and a stock.
  - The world is risk free after delta hedging.
  - Utility-free option pricing. Options are redundant.

- In a pure-diffusion world with stochastic volatility:
  - Market is complete with one (or a few) extra option(s).
  - The world is risk free after delta and vega hedging.

- In a world with jumps of random sizes:
  - The market is inherently incomplete (with stocks alone).
  - Need all options (+ model) to complete the market.
  - **Challenges**: Greeks-based dynamic hedging is no longer risk proof.
  - **Opportunities**: Options market is informative/useful:
    - Cross sections \((K, T) \Leftrightarrow \mathbb{Q} \) dynamics.
    - Time series \((t) \Leftrightarrow \mathbb{P} \) dynamics.
    - The difference \(\mathbb{Q}/\mathbb{P} \Leftrightarrow \) market prices of economic risks.
Beyond Lévy processes

- Lévy processes can generate different iid return innovation distributions.
  - Any distribution you can think of, we can specify a Lévy process, with the increments of the process matching that distribution.

- Yet, return distribution is not iid. It varies over time.
  - That’s why I have shown you only cross-sectional plots ...

- We need to go beyond Lévy processes to capture the time variation in the return distribution (implied volatility surface):
  - Stochastic volatility
  - Stochastic risk reversal (skewness)
  - Predictability of return or volatility.
At-the-money implied volatilities at fixed time-to-maturities from 1 month to 5 years.
Three-month delta-neutral straddle implied volatility.
Implied volatility spread between 80% and 120% strikes at fixed time-to-maturities from 1 month to 5 years.
Stochastic skewness on currencies

Three-month 10-delta risk reversal (blue lines) and butterfly spread (red lines).
Randomize the time

- Review the Lévy-Khintchine Theorem:
  \[
  \phi(u) \equiv \mathbb{E}[e^{iuX_t}] = e^{-t\psi(u)},
  \]
  \[
  \psi(u) = -iu\mu + \frac{1}{2}u^2\sigma^2 + \lambda \int_{\mathbb{R}_0} (1 - e^{iux} + iux\mathbf{1}_{|x|<1}) \tilde{\pi}(x)dx,
  \]

- The drift \(\mu\), the diffusion variance \(\sigma^2\), and the mean arrival rate \(\lambda\) are all proportional to time \(t\).

- We can directly specify \((\mu_t, \sigma^2_t, \lambda_t)\) as following stochastic processes.

- Or we can randomize time \(t \rightarrow T_t\) for the same result.

- We define \(T_t \equiv \int_0^t \nu_s^- ds\) as the **stochastic time change**, with \(\nu_t\) being the instantaneous activity rate.

- Depending on the Lévy specification, the activity rate has the same meaning (up to a scale) as a randomized version of the instantaneous drift, instantaneous variance, or instantaneous arrival rate.
In 1949, Bochner introduced the notion of time change to stochastic processes. In 1973, Clark suggested that time-changed diffusions could be used to accurately describe financial time series.

Ane & Geman (2000) show supporting evidence: Define returns over fixed number of trades, not over fixed calendar time intervals.

Two types of clocks can be used to model business time:

1. Clocks based on increasing jump processes have staircase like paths.
2. Continuous clocks ($T_t \equiv \int_0^t \nu_s \, ds$) as we have just defined.

The first type of clock can transform a diffusion into a jump process — All Lévy processes considered earlier can be generated as changing the clock of a diffusion with an increasing jump process (subordinator).

The second type of business clock can be used to describe stochastic volatility (and higher moments).

Monroe (1978): All semimartingales can be generated by applying stochastic time changes (of both types) on Brownian motions.
Economic interpretations

- Treat $t$ as the calendar time, and $\mathcal{T}_t \equiv \int_0^t v_s \, ds$ as the business time.
  - Business activity accumulates with calendar time, but the speed varies, depending on the business activity.
  - Business activity tends to intensify before earnings announcements, FOMC meeting days...
  - In this sense, $v_t$ captures the intensity of business activity at time $t$.
  - This interpretation has inspired many microstructure works...

- Economics shocks (impulse) and financial market responses:
  - Think of each Lévy process (component) as capturing one source of economic shock.
  - The stochastic time change on each Lévy component captures the random intensity of the impact of the economic shock on the financial security.

- \[
    \text{Return} \sim \sum_{i=1}^{K} X_{\mathcal{T}_t}^i \sim \sum_{i=1}^{K} (\text{Economic shock})^i_{\text{Stochastic impact}}.
\]
Example: Return on a stock

- Model the return on a stock to reflect shocks from two sources:
  
  - **Credit risk**: In case of corporate default, the stock price falls to zero. Model the impact as a Poisson Lévy jump process with log return jumps to negative infinity upon jump arrival.
  
  - **Market risk**: Daily market movements (small or large). Model the impact as a diffusion or infinite-activity (infinite variation) Lévy jump process or both.

- Apply separate time changes to the two Lévy components to capture (1) the intensity variation of corporate default, (2) the market risk (volatility) variation.

- **Key**: *Each component has a specific economic purpose.*

Example: A CAPM model:

\[ \ln \frac{S_t^j}{S_0^j} = (r - q)t + \left( \beta^j X^m_{T^m_t} - \varphi_x^m(\beta^j) T^m_t \right) + \left( X^j_{T^j_t} - \varphi_x^j(1) T^j_t \right). \]

- Estimate \( \beta \) and market prices of return and volatility risk using index and single name options.
- Cross-sectional analysis of the estimates.

An international CAPM:

Example: Return on an exchange rate

- Exchange rate reflects the interaction between two economic forces.
- Use two Lévy processes to model the two economic forces separately.
- Consider a negatively skewed distribution (downside jumps) from each economic source (crash-o-phobia from both sides). Use the difference to model the currency return between the two economies.
- Apply separate time changes to the two Lévy processes to capture the strength variation (tug war) between the two economic forces.
  - Stochastic time changes on the two negatively skewed Lévy processes generate both stochastic volatility and stochastic skew.
- Key: Each component has its specific economic purpose.

Example: Exchange rates and pricing kernels

- Exchange rate reflects the interaction between two economic forces.
- The economic meaning becomes clearer if we model the pricing kernel of each economy.
- Let $m_{0,t}^{US}$ and $m_{0,t}^{JP}$ denote the pricing kernels of the US and Japan. Then the dollar price of yen $S_t$ is given by
  \[
  \ln S_t / S_0 = \ln m_{0,t}^{JP} - \ln m_{0,t}^{US}.
  \]
- If we model the negative of the logarithm of each pricing kernel ($-\ln m_{0,t}^j$) as a time-changed Levy process, $X_{T_t}^j$ ($j = US, JP$) with negative skewness. Then, $\ln S_t / S_0 = \ln m_{0,t}^{JP} - \ln m_{0,t}^{US} = X_{T_t}^{US} - X_{T_t}^{JP}$
  - Think of $X$ as consumption growth shocks
  - Think of $T_t$ as time-varying risk premium.
- Consistent and simultaneous modeling of all currency pairs.

To compute the time-0 price of a European option price with expiry at $t$, we first compute the Fourier transform of the log return $s_t \equiv \ln S_t / S_0$.

The generalized Fourier transform of a time-changed Lévy process:

$$ \phi_Y(u) \equiv \mathbb{E}^Q \left[ e^{iuX_t} \right] = \mathbb{E}^M \left[ e^{-\psi_x(u)T_t} \right], \quad u \in D \subseteq \mathbb{C}, $$

where the new measure $\mathbb{M}$ is defined by the exponential martingale:

$$ \frac{d\mathbb{M}}{d\mathbb{Q}} \bigg|_t = \exp \left( iuX_t + T_t\psi_x(u) \right). $$

Without time-change, $e^{iuX_t + t\psi_x(u)}$ is an exponential martingale by Lévy-Khintchine Theorem.

A continuous time change does not change the martingality.

$\mathbb{M}$ is complex valued (no longer a probability measure).

Tractability of the transform $\phi(u)$ depends on the tractability of

- The characteristic exponent of the Lévy process $\psi_x(u)$.
- The Laplace transform of $T_t$ under $\mathbb{M}$.

$(X, T_t)$ can be chosen separately as building blocks to capture the two dimensions: Moneyness & term structure.
The Laplace transform of the stochastic time $\mathcal{T}_t$

- We have solved the characteristic exponent of the Lévy process (by the Lévy-Khintchine Theorem).
- Compare the Laplace transform of the stochastic time,

$$\mathcal{L}_\mathcal{T}(\psi) \equiv \mathbb{E} \left[ e^{-\psi T_t} \right] = \mathbb{E} \left[ e^{-\psi \int_0^t \nu_s ds} \right]$$

(1)

- to the pricing equation for zero-coupon bonds:

$$B(0, t) \equiv \mathbb{E}^Q \left[ e^{-\int_0^t r_s ds} \right]$$

(2)

- The two pricing equations look analogous
  - Both $\nu_t$ and $r_t$ need to be positive.
  - If we set $r_t = \psi \nu_t$, $\mathcal{L}_\mathcal{T}(\psi)$ is essentially the bond price.

- The analogy allows us to borrow the vast bond pricing literature:
  - **Affine class**: Zero-coupon bond prices are exponential affine in the state variable.
  - **Quadratic**: Zero-coupon bond prices are exponential quadratic in the state variable.

...
With the Fourier transform of the log return ($\phi(u)$), we can compute vanilla option values via Fourier inversion.

Take a European call option as an example.

Perform the following rescaling and change of variables:

$$c(k) = e^{rt} c(K, t)/F_0 = \mathbb{E}_0^Q \left[(e^{s_t} - e^k)1_{s_t \geq k}\right],$$

with $s_t = \ln F_t/F_0$ and $k = \ln K/F_0$.

- $c(k)$: the option forward price in percentage of the underlying forward as a function of moneyness defined as the log strike over forward, $k$ (at a fixed time to maturity).

Derive the Fourier transform of the scaled option value $c(k)$ ($\chi_c(u)$) in terms of the Fourier transform ($\phi_s(u)$) of the log return $s_t = \ln F_t/F_0$.

Perform numerical Fourier inversion to obtain option value.

There are two ways of doing this.
I. The CDF analog

- Treat $c(k) = \mathbb{E}_0^Q \left[ (e^{s_t} - e^k) 1_{s_t \geq k} \right] = \int_{-\infty}^{\infty} (e^{s_t} - e^k) 1_{s_t \geq x} dF(s)$ as a CDF.

- The option transform:

$$\chi^I_c(u) \equiv \int_{-\infty}^{\infty} e^{iuk} dc(k) = -\frac{\phi_s(u - i)}{iu + 1}, \quad u \in \mathbb{R}.$$ 

- The inversion formula is analogous to the inversion of a CDF:

$$c(x) = \frac{1}{2} + \frac{1}{2\pi} \int_{0}^{\infty} \frac{e^{iux} \chi^I_c(-u) - e^{-iux} \chi^I_c(u)}{iu} du.$$

- Use quadrature methods for the numerical integration.
  - It can work well if done right.

  
II. The PDF analog

- Treat \( c(k) \) analogous to a PDF. (Carr and Madan (1999), Carr & Wu (2004), ...)

  - The option transform:

    \[
    \chi_{c}^{II}(z) \equiv \int_{-\infty}^{\infty} e^{ikz} c(k) dk = \frac{\phi_s (z - i)}{(iz)(iz + 1)}
    \]

    with \( z = u - i\alpha \), \( \alpha \in \mathcal{D} \subseteq \mathbb{R}^{+} \) for the transform to be well defined.

    - The range of \( \alpha \) depends on payoff structure and model.
    - The exact choice of \( \alpha \) is a numerical issue (asking for more research...)

  - The inversion is analogous to that for a PDF:

    \[
    c(k) = \frac{1}{2} \int_{-i\alpha - \infty}^{-i\alpha + \infty} e^{-ikz} \chi_{c}^{II}(z) dz = \frac{e^{-\alpha k}}{\pi} \int_{0}^{\infty} e^{-iu(k)} \chi_{c}^{II}(u - i\alpha) du.
    \]

    - The numerical integration can be cast into an FFT to improve the computational speed.

    - Use fractional FFT to separate the choice of strike grids from the integration grids (Chourdakis (2005)).

      - Room for improvement...
Estimating statistical dynamics

  - Given initial parameters guess, derive the return characteristic function.
  - Apply FFT to generate the probability density at a fine grid of possible return realizations.
  - Interpolate to obtain the density at the observed return values.
  - Numerically maximize the aggregate log likelihood.

- For time-changed Lévy processes with observable activity rates, it is still straightforward to apply MLE.

- For time-changed Lévy processes with hidden activity rates, some filtering technique is needed to infer the hidden states from the observable.
  - Maximum likelihood with partial filtering: Alireza Javaheri
  - MCMC Bayesian estimation: Eraker, Johannes, Polson (2003, JF), Li, Wells, Yu, (RFS)

- Use more data (and transformation) to turn hidden states into observable quantities. Wu (2007), Aït-Sahalia and Robert Kimmel (2007), Bondarenko (2007)…
Estimating risk-neutral dynamics

- **Daily fitting**: (Bakshi, Cao, Chen (1997, JF), Carr and Wu (2003, JF))
  - Nonlinear weighted least square to fit models to option prices.
  - Parameters and state variables (activity rates) are treated as the same.
  - *What to hedge*: state variables or parameters or both.
  - Can experience identification issues for sophisticated models.
  - Better applied to Lévy processes without time change.

- **Dynamically consistent estimation**:
  - Parameters are fixed, only activity rates are allowed to vary over time.
  - Numerically more challenging.
  - Better applied to more sophisticated models that perform well over different market conditions.
Static v. dynamic consistency

- **Static cross-sectional consistency**: Option values across different strikes/maturities are generated from the same model (same parameters) at a point in time.

- **Dynamic consistency**: Option values over time are also generated from the same no-arbitrage model (same parameters).

Different needs for different market participants:

- **Market makers**:
  - Achieving static consistency is sufficient.
  - Matching market prices is important to provide two-sided quotes.

- **Long-term convergence traders**:
  - Dynamic consistency is important.
  - A good model should generate large (you wish) but highly convergent pricing errors, and provide robust hedging ratios.

A well-designed model (with several time-changed Lévy components) can achieve both dynamic consistency and good performance.
Dynamically consistent estimation

- Nested nonlinear least square (Huang & Wu (2004), Bates (2000)):
  Often has convergence issues.

- Cast the model into state-space form and use MLE
  (Carr & Wu (2007a,b), Bakshi, Carr, Wu (2008), Mo& Wu (2007), Heidari & Wu (2008), Leippold & Wu (2007),...)
  - Define state propagation equation based on the \( \mathbb{P} \)-dynamics of the activity rates. (Need to specify market price on activity rates).
  - Define the measurement equation based on option prices (out-of-money values, weighted by vega,...)
  - Use an extended version of Kalman filter (EKF, UKF, PKF) to predict/filter the distribution of the states and measurements.
  - Define the likelihood function based on forecasting errors on the measurement equations.
  - Estimate model parameters by maximizing the likelihood.
Joint estimation of $P$ and $Q$ dynamics

*Important in learning investors’ risk-taking behaviors.*

- Pan (2002): GMM.
- Eraker (2004): Bayesian with MCMC. Choose 2-3 options per day!
- Bakshi & Wu (2005), “Investor Irrationality and the Nasdaq Bubble”
  MLE with filtering
  - Cast activity rate $P$-dynamics into state equation, cast option prices into measurement equation.
  - Use UKF to filter out the mean and covariance of the states and measurement.
  - Construct the likelihood function of options based on forecasting errors (from UKF) on the measurement equations.
  - Given the filtered activity rates, construct the conditional likelihood on the Nasdaq-100 index returns by FFT inversion of the conditional characteristic function.
  - The joint log likelihood equals the sum of the log likelihood of option pricing errors and the conditional log likelihood of index returns.
Concluding remarks

- Modeling security returns with time-changed Lévy processes enjoys three key virtues: (1) Generality; (2) explicit economic mapping; (3) tractability.

- The framework provides a nice starting point for generating security return dynamics that are parsimonious, tractable, economically sensible, and statistically performing well.

- It offers many research opportunities... and challenges:
  - Measure theory related to Lévy jumps.
  - Refine (fractional) FFT to generate the density and option values at the relevant region.
  - Solving multi-dimensional PIDEs in the presence of Lévy jumps.
  - Robust and efficient simulation of Lévy jump processes.
  - Embed simulation into model estimation (MCMC, particle filter, pricing under new models, for exotics).