Advanced Bond Pricing

An Example of Bond Arbitrage

- Three bonds A, B and C have face values of $1000
  - A is a one-year zero coupon bond with a current price of $900
  - B is a two-zero zero coupon bond with a current price of $800
  - C is a two-year coupon bond with an annual coupon of 10% and a current price of $1000

Q: Identify the arbitrage opportunity and design a trading strategy to exploit the mis-pricing
Bond Discount Factors

- Discount factor \((d_t)\) defines the present value of a dollar to be paid in \(t\) years.
- A coupon bond can be viewed as a portfolio of zero-coupon bond.
  - A coupon bond can be priced as the sum of the \(PV\) of its future coupon and par payments using relevant discount factors.

\[
P_{\text{PV}} = \sum_{t=1}^{T} d_t c_t
\]

Spot Rate

- \(YTM\) on a pure discount security (zero coupon security).
- Spot rate equation:

\[
P_{0,t} = \frac{F_t}{(1 + s_t)^t}
\]

Where \(P_{0,t}\) = the current market price (at time \(t = 0\)) of a pure discount bond maturing in \(t\) years.
- \(F_t\) = the face value of the zero-coupon bond.
- \(s_t\) = the spot rate.

- We can price a bond by discounting different coupons and par value by their corresponding spot rates.
An Example

- What is the relationship between discount factor and bond spot rates?

- Q: What is $d_1$ and $d_2$ as implied in bond A and bond B?

- Q: Based on $d_1$ and $d_2$, what should be an appropriate price for bond C?

- Q: How can you take advantage of mis-pricing?

Forward Rate

- The interest rate decided today that will be paid on money to be borrowed at some specific future date and to be repaid at a specific more distant future date

- Notation: $f_{1,2}$ is the interest rate agreed today on a one-year loan (to be) made in one year from now and matures in two years from today
Spot Rates vs. Forward Rates

- Link between 1-year spot rate, 2-year spot rate and 1-year forward rate

\[
(1 + f_{1, 2}) = \frac{(1 + s_2)^2}{1 + s_1}
\]

- In general,

\[
(1 + f_{t-i,t}) = \frac{(1 + s_t)^{t-i}}{(1 + s_{t-i})^{t-1}}
\]

Example

- $1 paid in one year has a PV of 0.9346 and $1 paid in two years has a PV of 0.8573,

- Q: What is \(d_1\), \(d_2\), \(s_1\), \(s_2\), and \(f_{1,2}\)?

\[
d_1 = 0.9346 = \frac{1.0}{1 + s_1}
\]

\[
d_2 = 0.8573 = \frac{1.0}{(1 + s_2)^2}
\]

\[
(1 + s_1)(1 + f_{1,2}) = (1 + s_2)^2
\]
Spot Rates, Forward Rates and Discount Factor

- **Spot rate**
  \[ d_t = \frac{1}{(1 + s_t)^t} \Rightarrow s_t = \left[ \frac{1}{d_t} \right]^{\frac{1}{t}} - 1 \]

- **Forward rate**
  \[ (1 + s_t)^t = (1 + s_{t-1})^{t-1} (1 + f_{t-1,t}) \]
  \[ 1 + f_{t-1,t} = \frac{(1 + s_t)^t}{(1 + s_{t-1})^{t-1}} = d_{t-1}/d_t \]

Wrap-up

- **Discount factor**
- **Spot rate**
- **Forward rate**
- **Bond arbitrage-free pricing**