Informed Traders, News and Volatility*

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Abstract

We define news as a sequence of private signals received by a set of traders about an asset value that is moving over time. Usually, in continuous time market microstructure models of asymmetric information, informed traders only contribute to the drift of price changes, while the volatility component is generated entirely by the noise traders. In contrast, we show that in the presence of news, competition among informed traders adds a stochastic component to their trading, and hence contribute to an informed component of volatility.

Keywords: Insider trading, Kyle model, noise trading, stochastic volatility.

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1 Introduction

The literature on asset price volatility has shown that a significant component of volatility is due to news. Yet, since the landmark paper of Kyle (1985), the literature on price impact under asymmetric information has shown that the instantaneous price volatility due to informed traders is very small compared to the price volatility due to noise traders. Indeed, in the continuous version of the model in Kyle (1985), as well as in Back (1992), or Back, Cao, and Willard (2000), informed traders contribute to prices by adding only a drift component, while the volatility component of price changes is generated only by noise traders.

But why would noise traders add volatility to markets? In an influential paper, Black (1986) writes that “noise trading is trading on noise as if it were information.” In other words, the reason noise traders generate volatility is that they behave like informed traders. However, if informed traders do not themselves generate volatility, then the question remains as to why there is volatility in markets in the first place. Our paper provides a solution to the problem. In the presence of a moving fundamental value, competition among informed traders generates volatility.

The asset pricing literature typically has solved this problem by studying volatility and trading volume in a noisy rational expectations equilibrium (REE) context, e.g., see Wang (1993). In this literature, the informed traders’ demand has a stochastic component, but this is because traders are price takers and do not internalize the impact of their trades on prices.

Our paper shows that, under a set of conditions, informed traders facing the price impact of their trading optimally submit orders with a stochastic component, and thus contribute to an informed component of volatility. The first condition is that the asset’s fundamental value moves randomly over time, which we interpret as coming from the influence of news. A large part of the market microstructure literature assumes, along with Kyle (1985), that the fundamental (or liquidation) value is constant. We argue that the presence of news about, e.g., the productivity of a firm, or its competitive position suggests that in reality the liquidation value changes over time.

1See, e.g., Roll (1988) for equity volatility, and Ederington and Lee (1993) for interest rate and foreign exchange volatility.
The second condition is that there are at least two informed traders. We show that without competition, in continuous time an informed trader does not use his private signal about the next instant’s fundamental value. This is similar to the result of Back (1992), which in a constant fundamental value setup shows that a monopolist insider would not include a stochastic component in his order (see Lemma 2 and the subsequent discussion). The intuition behind this condition is also related to Holden and Subrahmanyam (1992). They show that competing informed traders who observe the same (constant) fundamental value would reveal their information infinitely fast in continuous time. Our paper shows that informed traders who observe the same signal about a moving fundamental value would not trade infinitely fast, but instead would incorporate the stochastic signals in their orders. Trading infinitely fast with a stochastic component would indeed add too much to the cost of trading.

The third condition is that there exists enough noise trading to support the equilibrium. As usual, we need noise traders to avoid the no-trade theorem of Milgrom and Stokey (1982). But unlike the case of Kyle (1985), where any amount of noise trading is sufficient to support the equilibrium, in this paper we need the amount of noise trading to be above a threshold.

One of the benefits of having informed traders submit stochastic orders is a better understanding of stochastic volatility, and its connection with price formation. If informed traders do not contribute to instantaneous price volatility, then one can only obtain stochastic price volatility by exogenously assuming a stochastic noise trading component. If, instead, informed traders submit stochastic orders, then their presence or absence from the market would generate stochastic volatility. One possible application is to the multifractal volatility model of Calvet and Fisher (2001). There, the multifractal aspect of volatility can be interpreted as resulting from informed traders who acquire information and enter or exit the market at certain frequencies.

Our paper points to an alternative approach to the literatures of differences in higher-order beliefs and forecasting the forecasts of others.\footnote{See Foster and Viswanathan (1996), Bernhardt, Seiler, and Taub (2009), Banerjee, Kaniel, and Kremer (2009), and Banerjee and Kremer (2010).} We assume in this paper that informed traders have identical signals about the fundamental value. Otherwise, informed
traders would have to forecast the signals of other traders, and the informed traders
would have to use the whole past history of prices and signals. The alternative is to
have informed traders forget their past signals, and only use one summary statistic for
the fundamental value, even when forecasting the signals of others. If the loss in ex-
pected utility from forgetting the history is not large, we can think of these strategies
as near-rational. The advantage of a near-rational approach is that one could repro-
duce the results from the literature of differences in beliefs, and without undergoing the
complications of forecasting the forecasts of others.

Besides obtaining an informed component of volatility, our paper has a methodologi-
cal contribution. Unlike Back (1992) or Back, Cao, and Willard (2000), whose approach
involves guessing the solution and using the Hamilton-Jacobi-Bellman function, we use
a variational calculus approach that provides a constructive way of searching for the
solution.

The paper is organized as follows. Section 2 describes the model. Section 3 describes
the resulting equilibrium price process and trading strategies. Section 4 concludes.

2 The Model

Trading takes place in continuous time, with time in the interval \([0, 1]\). To describe
the sources of uncertainty in the model, consider three independent Brownian motion
processes, \(B_j\) with \(j = 1, 2, 3\). There is a single risky asset with liquidation value, or fun-
damental value, \(v_t\), which follows a Brownian motion with zero drift, and instantaneous
volatility \(\sigma_v\), i.e.,

\[
dv_t = \sigma_v\, dB_{1,t}. \tag{1}
\]

There are three types of market participants: noise traders, informed traders, and market
makers. Noise traders trade an exogenous amount \(du_t\) at each \(t\), given by

\[
du_t = \sigma_u\, dB_{2,t}. \tag{2}
\]
where we allow noise trading volatility \( \sigma_{u,t} \) to be time dependent.\(^3\) There are \( N \) risk-neutral informed traders who learn about the fundamental value of the asset. They all observe the same information. Initially, they get a noisy signal \( s_0 \) about \( v_0 \), with ex ante variance \( \Sigma_0 \). Subsequently, informed traders receive a stochastic information flow: immediately before trading at \( t \), they observe a noisy signal \( d s_t \) about the change in fundamental value, \( d v_t \),

\[
    d s_t = d v_t + d \eta_t, \quad \text{with} \quad d \eta_t = \sigma_\eta \, dB_{3,t},
\]

(3)

such that the volatility \( \sigma_\eta \) is constant. Note that \( \sigma_\eta = 0 \) implies that the informed traders perfectly observe the fundamental value \( v_t \) at all times. After trading at \( t \), the position of informed trader \( j \) in the asset is \( x_{t}^j \). Assume that the asset liquidates at \( t = 1 \). Then the expected future profit of informed trader \( j \) at \( t \) is given by

\[
    \pi_{s,t}^j = \mathbb{E}_t^s \int_t^1 (v_1 - P_\tau - dP_\tau) \, dx_{\tau}^j, \quad j = 1, \ldots, N,
\]

(4)

where \( \mathbb{E}_t^s \) indicates expectation conditional on the public information available at \( t \) (order flow and prices until \( t \)), and the private signals \( s_\tau \) for \( \tau \in [0, t] \). In the Kyle (1985) model, the \( dP_s \) component is omitted, since it is of the order of \( dt \). Here, both the price change \( dP_s \) and the market order \( dx_s \) potentially have a stochastic component, and thus we cannot ignore the term \( dP_s \, dx_s \). This reflects the fact that informed traders take into account their price impact when they trade.

Finally, risk-neutral competitive market makers observe the aggregate trading at \( t \),

\[
    d y_t = d u_t + \sum_{j=1}^{N} dx_{t}^j, \quad (5)
\]

or equivalently they observe the aggregate positions, \( y_t = u_t + \sum_{j=1}^{N} x_{t}^j \). Denote by \( \mathcal{F}_t \) the filtration generated by the order flow, which is public information. The market

\(^3\) We also allow \( \sigma_u \) to depend on other parameters of the model, such as the price impact coefficient, \( \lambda_t \) (see below).
makers determine the price at \( t \), conditional on the past order flow,

\[
P_t = E(v_1 | \mathcal{F}_t). \tag{6}
\]

The order of events is as follows. At \( t \), the informed traders observe the signal \( d s_t \) about the change in the fundamental value, \( d v_t \), and separately submit a market order of size \( d x^j_t \) to a brokerage. Independently, the noise traders submit an aggregate market order, \( d u_t \). The brokerage aggregates all the orders and submits the net result as a market order to a set of competitive market makers. Thus, the market makers observe the aggregate market order, \( d y_t \), and set the price \( P_{t+dt} = P_t + dP_t \), at which the trading takes place. The price change, \( dP_t \), takes into account information contained in the aggregate order flow, \( d y_t \).

### 3 Equilibrium

The description of the equilibrium concept is similar to that of Back, Cao and Willard (2000). The novel feature is the existence of a volatility component of informed trading. We are looking for an equilibrium in which informed traders submit demands linear in the signal,

\[
dx^j_t = \gamma^j dt + \mu^j d s_t, \quad j = 1, \ldots, N, \tag{7}
\]

and the price changes are linear in the net demand changes,

\[
dP_t = \lambda_t d y_t. \tag{8}
\]

The price impact coefficient, \( \lambda_t \), (also called the “Kyle lambda”) is a measure of the liquidity of the market. A large \( \lambda_t \) implies that market orders move the price by a lot, i.e., the market is illiquid.

The next result provides necessary conditions for the existence of a linear equilibrium. It shows that in the absence of competition among informed traders (\( N = 1 \)), the order of the monopolist insider does not have a volatility component, i.e., \( \mu^j_t = 0 \). With more than one trader (\( N \geq 2 \)), \( \mu^j_t > 0 \), i.e., their orders have a nonzero volatility component.
Recall that $\mathcal{F}_t$ is the filtration generated by the public information available at time $t$ (past prices and aggregate order flow). Denote by $\mathcal{F}_t^s$ the filtration generated by the information held by each informed trader (they get identical signals), i.e., $\mathcal{F}_t^s$ is the filtration generated by the public information $\mathcal{F}_t$, together with all the signals observed by informed traders until $t$. Define

$$w_t^s = \mathbb{E}(v_1|\mathcal{F}_t^s), \quad (9)$$

$$\Sigma_t = \text{Var}(w_t^s|\mathcal{F}_t). \quad (10)$$

The variable $w_t^s$ represents the filtered fundamental value, $v_1$, from the point of view of the informed traders. Define also

$$\nu = \frac{N^2 - N}{N^2 + 1}, \quad (11)$$

$$a = \frac{\sigma_v^2}{\sigma_v^2 + \sigma_\eta^2}. \quad (12)$$

Recall that by linear equilibrium we mean that the optimal strategy of the informed traders are of the form $dx_t = \gamma_t \, dt + \mu_t \, ds_t$, and the price change is of the form $dP_t = \lambda_t \, dy_t$, where $dy_t$ is the change in the aggregate demand. The next result gives necessary conditions for the existence of a linear equilibrium. In particular, with a monopolist informed trader ($N = 1$), the informed trader strategy must have $\mu_t = 0$, i.e., the optimal strategy has no volatility component. With competition ($N \geq 2$), the optimal strategy has $\mu_t > 0$, i.e., there is an information component of volatility.

**Theorem 1.** Suppose there exists a linear equilibrium as defined above. If $N = 1$, assume further that the informed strategies have the drift coefficient of the form $\gamma_t = \beta_t (w_t^s - P_t)$. If $N \geq 2$, then any optimal strategy must have this form. The following are necessary conditions for the existence of a linear equilibrium:

- If $N = 1$

$$\lambda_t = \lambda = \text{constant}, \quad (13)$$

$$\int_t^1 \beta_\tau \, d\tau = +\infty. \quad (14)$$
• If \( N \geq 2 \)

\[
\lambda_t \to 0 \quad \text{when} \quad t \to 1, \quad (15)
\]

\[
\beta_t = -\frac{1}{N-1} \frac{\lambda_t'}{\lambda_t^2}. \quad (16)
\]

• For all \( N \)

\[
\mu_t = \frac{N-1}{N^2 + 1} \frac{a}{\lambda_t} = \frac{aN}{N\lambda_t}, \quad (17)
\]

\[
\Sigma_t = \frac{\sigma_{u,t}^2 \lambda_t^2 - a\sigma_v^2 (\nu - \nu^2)}{N\beta_t \lambda_t}, \quad (18)
\]

\[
\Sigma_t' = a\sigma_v^2 (1 - \nu^2) - \sigma_{u,t}^2 \lambda_t^2, \quad (19)
\]

\[
\int_0^1 \sigma_{u,t}^2 \lambda_t^2 \, dt = \Sigma_0 + a\sigma_v^2 (1 - \nu^2). \quad (20)
\]

When \( N \geq 2 \), the conditions above are also sufficient for the existence of a linear equilibrium.

\textit{Proof.} See the Appendix. \(\square\)

Note that if the informed traders see the fundamental value at all times \((\sigma_\eta = 0)\), then \(a = \frac{\sigma_v^2}{\sigma_v^2 + \sigma_\eta^2} = 1\), and the formulas are somewhat simpler.

The proof of this Theorem uses the calculus of variations, as described in, e.g., Gelfand and Fomin (2003). One advantage of this approach is that it provides a constructive way to search for the solution. In particular, Equations (18) and (19) imply the following differential equation:

\[
\left( \frac{\sigma_{u,t}^2 \lambda_t^2 - a\sigma_v^2 (\nu - \nu^2)}{N\beta_t \lambda_t} \right)' = a\sigma_v^2 (1 - \nu^2) - \sigma_{u,t}^2 \lambda_t^2. \quad (21)
\]

When \( N = 1 \), \( \lambda \) is constant, therefore (21) gives a first order differential equation in \( \beta_t \).

When \( N \geq 2 \), we use Equation (16), which says that \( \beta_t = -\frac{1}{N-1} \frac{\lambda_t'}{\lambda_t^2} \). In that case, (21) is a second order differential equation in \( \lambda_t \). Then Equation (20) can be used to show that the solution is unique.

When the number of informed traders is large, the constant \( \nu = \frac{N^2 - N}{N^2 + 1} \) is very close to 1, and the equation that \( \lambda \) approximately takes a simpler form:

\[
\left( \frac{\sigma_{u,t}^2 \lambda_t^3}{\lambda_t} \right)' = \sigma_{u,t}^2 \lambda_t^2,
\]
which can be rewritten as
\[
\frac{\lambda''_t}{\lambda_t} = \frac{2\sigma'_{u,t}}{\sigma_{u,t}^2} + \frac{2\lambda'_t}{\lambda_t}.
\] (22)

This equation can be solved in closed form, as long as the noise trading volatility is large enough.

**Proposition 1.** Suppose the noise trading volatility satisfies
\[
\int_0^1 \sigma_{u,\tau}^2 \, d\tau = +\infty.
\] (23)

If the number of informed traders is very large, then the price impact coefficient \(\lambda_t\) converges to the following:
\[
\frac{1}{\lambda_t} = \frac{1}{\lambda_0} + C \int_0^t \sigma_{u,\tau}^2 \, d\tau,
\] (24)

where the constant \(C\) is determined by
\[
\int_0^1 \frac{\sigma_{u,t}^2}{\left(\frac{1}{\lambda_0} + C \int_0^t \sigma_{u,\tau}^2 \, d\tau\right)^2} \, dt = \Sigma_0.
\] (25)

In particular, if the noise trading volatility \(\sigma_{u,t}\) has the form
\[
\sigma_{u,t} = \sigma_{u,0}(1 - t)^{-\alpha}, \quad \text{with} \quad \alpha > \frac{1}{2},
\] (26)

then \(\lambda_t\) converges to
\[
\lambda_t = \left(\frac{2\alpha - 1}{\frac{\Sigma_0}{\sigma_{u,0}^2}}\right)^{1/2} (1 - t)^{2\alpha - 1}.
\] (27)

**Proof.** See the Appendix. \(\square\)

Except for this limiting case, when \(N \geq 2\), in general we must use numerical methods to solve Equation (21). As before, the existence of a solution requires the amount of noise trading to be above a threshold. To see this, note that Theorem 1 requires that \(\lambda_t \to 0\) when \(t \to 1\), thus we should expect the trading intensity \(\beta_t = -\frac{\lambda'_t}{\lambda_t^2}\) to converge to infinity when \(t \to 1\). In order to support this high level of informed trading, it is reasonable to expect that the amount of noise trading is also very high as \(t\) approaches
1. We illustrate this situation by analyzing a particular equilibrium which can be solved in closed form. Recall that $\nu = \frac{N^2 - N}{N^2 + 1}$.

**Proposition 2.** Let $N \geq 2$. Suppose the volatility of noise trading has the form

$$
\sigma_{u,t} = \sigma_{u,0}(1 - t)^{-\alpha}, \quad \text{with} \quad \alpha = \frac{(N - 1)(3N - 2)}{2(2N^2 - N + 1)},
$$

Then there is a unique linear equilibrium. For this equilibrium,

$$
\lambda_t = \left( \frac{bN^N}{\sigma_{u,0}^2} \left( \sum_0 + a\sigma_v^2(1 - \nu^2) \right) \right)^{1/2} (1 - t)^b,
$$

$$
\beta_t = \left( \frac{b}{\Sigma_0 + a\sigma_v^2(1 - \nu^2)} \right)^{1/2} \frac{1}{(1 - t)^{b+1}},
$$

$$
\mu_t = \frac{a\nu}{N\lambda_t},
$$

$$
b = \frac{(N - 1)^2}{2N^2 - N + 1}.
$$

**Proof.** See the Appendix.

An alternative equilibrium concept can be obtained by assuming that the noise trading volatility is a function of the price impact coefficient. Indeed, as in Admati and Pfleiderer (1988), a more liquid market attracts more noise trading. The advantage of this approach, as the next result illustrates, is that the equilibrium can then be expressed in essentially closed form.

**Theorem 2.** Suppose there are $N \geq 2$ informed traders, and the noise trading volatility is a function of the price impact coefficient, i.e., $\sigma_{u,t} = \sigma_u(\lambda_t)$. The function $A(\lambda) = \sigma_u^2(\lambda)\lambda^2$ is continuous and satisfies

$$
A(\lambda) > a\sigma_v^2(1 - \nu^2) \quad \forall \lambda > 0,
$$

along with the following technical conditions: (i) there exist $\lambda_1, \lambda_2 > 0$ such that $A(\lambda_1) \leq \Sigma_0 + a\sigma_v^2(1 - \nu^2) \leq A(\lambda_2)$; and (ii) $A(\lambda)$ has does not grow faster than $\lambda^{\frac{N-1}{N}}$ when $\lambda \to \infty$. 

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Then there exists a linear equilibrium. For this equilibrium,

\[
\int_0^{\lambda} e^{-F(x)} \, dx = C(1 - t), \tag{35}
\]

\[
F(\lambda) = \int_{\lambda_0}^{\lambda} \frac{A'(x)x + \frac{1}{N} A(x) + a\sigma^2_v \left( \frac{N-1}{N} (1 - \nu^2) - (\nu - \nu^2) \right)}{x(A(x) - a\sigma^2_v (\nu - \nu^2))} \, dx. \tag{36}
\]

The constant \(C\) is determined by

\[
\int_0^1 A(\lambda_t) \, dt = \Sigma_0 + a\sigma^2_v (1 - \nu^2). \tag{37}
\]

**Proof.** See the Appendix. \qed

With this equilibrium concept, multiple equilibria can arise. This is because, as in Admati and Pfleiderer (1988), informed trading and noise trading are strategic complements. More intense informed trading activity leads to quicker revelation of information, and hence to more noise trading activity, which in turn implies that informed traders can better hide behind noise trading, and therefore trade more intensely.

In a particular case of the above Theorem, we can also estimate what percentage the variance of informed trading is of total variance. Recall that in a symmetric linear equilibrium with \(N\) informed traders, the infinitesimal price change has the form

\[
dP_t = \lambda_t dy_t = \lambda_t (\sigma_{u,t} \, du_t + N\gamma_t \, dt + N\mu_t \, ds_t). \tag{38}
\]

The noise component is \(\sigma^2_{u,t} \lambda_t^2 \, dt\), and the informed component is \((N\mu_t \lambda_t)^2 (\sigma^2_v + \sigma^2_\eta) \, dt\). Therefore, we define the informed variance ratio to be the number \(\rho_t\) given by

\[
\rho_t = \frac{(N\mu_t \lambda_t)^2}{\sigma^2_{u,t} \lambda_t^2 + (N\mu_t \lambda_t)^2}. \tag{39}
\]

The next result shows that the informed variance ratio must be below a threshold, i.e., which is another way of expressing the intuition that in order for an equilibrium to exist, the level of noise trading must be above a threshold. Recall that \(\nu = \frac{N^2 - N}{N^2 + 1}\).

**Corollary 1.** In the setup of Theorem 2, suppose the function \(A(\lambda) = \sigma^2_{u,t}(\lambda) \lambda^2\) is constant, and \(A(\lambda) = \Sigma_0 + a\sigma^2_v (1 - \nu^2)\). Then there exists a family of linear equilibria
parametrized by a constant \( C > 0 \), for which the price impact coefficient is

\[
\lambda_t = C(1-c)(1-t)^{1/2}, \quad \text{with} \quad c = \frac{1}{N} + \frac{N-1}{N \left( 1 + \frac{\Sigma_0}{a \sigma_v^2 (1-\nu)} \right)} < 1.
\] (39)

For each equilibrium in this family, the informed variance ratio is constant:

\[
\rho_t = \frac{\nu^2}{\frac{\Sigma_0}{a \sigma_v^2} + 1}.
\] (40)

This implies that \( \rho_t < \nu^2 \) for all \( t \in [0,1] \).

**Proof.** See the Appendix. \qed

Next, we analyze the case in which there is only one informed trader. We revert to the situation in which the volatility of noise trading, \( \sigma_{u,t} \), is exogenously specified.\(^4\) The case \( N = 1 \) is special, in that there are many optimal strategies of the monopolist insider that lead to the same expected utility (see, e.g., Back (1992)). But, if we restrict our attention to equilibria in which the informed trader submits orders for which the drift component is of the form \( \gamma_t \, dt = \beta_t (w_t^d - P_t) \, dt \), the equilibrium is unique.

**Proposition 3.** If \( N = 1 \), assume that the noise trading volatility \( \sigma_{u,t} \) satisfies the following technical condition: for all \( t \in [0,1] \),

\[
\int_0^1 \frac{\sigma_{u,t}^2 \, d\tau}{1-t} > \frac{a \sigma_v^2}{a \sigma_v^2 + \Sigma_0} \int_0^1 \sigma_{u,t}^2 \, d\tau.
\]

Then there is a unique linear equilibrium in which the informed trader has a strategy of the type

\[
dx_t = \beta_t (w_t^d - P_t) \, dt + \mu_t \, ds_t.
\]

For this equilibrium,

\[
\lambda_t = \lambda = \left( \frac{a \sigma_v^2 + \Sigma_0}{\int_0^1 \sigma_{u,t}^2 \, d\tau} \right)^{1/2},
\] (41)

\[
\beta_t = \frac{1}{\lambda (1-t)} \frac{\sigma_{u,t}^2}{\frac{\sigma_{u,t}^2}{1-t} - \frac{a \sigma_v^2}{a \sigma_v^2 + \Sigma_0} \int_0^1 \sigma_{u,t}^2 \, d\tau},
\] (42)

\[
\mu_t = 0.
\] (43)

**Proof.** See the Appendix. \qed

The technical condition is true, e.g., if the function \( \sigma_{u,t} \) is weakly monotonically increasing over \([0,1]\), which includes the case when \( \sigma_{u,t} \) is constant. If \( \sigma_{u,t} = \sigma_u \) is

\(^4\)The discussion below applies also to the the case when \( \sigma_{u,t} \) is a function of \( \lambda \).
constant, we compute

\[ \lambda = \frac{\sqrt{\Sigma_0 + a\sigma_v^2}}{\sigma_n}, \quad (44) \]

\[ \beta_t = \frac{\sigma_u \sqrt{\Sigma_0 + a\sigma_v^2}}{\Sigma_0} \frac{1}{1-t}. \quad (45) \]

If the noise trading volatility is constant, if the fundamental value is constant \((\sigma_v = 0)\), and if the informed trader has a perfect signal about the fundamental value, then the previous Proposition reduces to the continuous time version of Kyle (1985) model with a monopolist insider.

4 Conclusions

We have shown that in the presence of news, i.e., when the fundamental value is moving over time, competing informed traders optimally add a stochastic component to their orders, hence bring an informed component to volatility and trading volume. This stands in contrast with traditional market microstructure models, such as Kyle (1985), or Back, Cao, and Willard (2000), who show that informed traders only contribute to the drift, but not to the volatility component of price changes.

Since informed traders create volatility by their orders, one can potentially use this paper to infer the structure of informed trading by analyzing the volatility process. In particular, the multifractal volatility model of Calvet and Fisher (2001) can be interpreted as coming from the contribution of informed traders entering and exiting the market at various frequencies. Also, this paper provides an alternative to the literature of differences in beliefs, by suggesting a near-rational model. If traders only use part of their signals to forecast the forecast of others, one would perhaps obtain similar effects on volatility and trading volume that Banerjee, Kaniel and Kremer (2009), and Banerjee and Kremer (2010) obtain in a setup involving differences in higher-order beliefs.
A Proofs of Results

Proof of Theorem 1: Recall that all informed traders have the same information: an initial signal $s_0$, and an information flow $d s_t = d v_t + d \eta_t$, with $d \eta_t = \sigma_\eta \, d B_{3,t}$. The filtered fundamental value, $w_t^i = E(v_1 | \mathcal{F}_t^i)$, evolves according to:

\begin{align}
    w_0 &= s_0, \quad (46) \\
    d w_t^i &= a \, d s_t, \quad \text{with} \quad a = \frac{\sigma_v^2}{\sigma_v^2 + \sigma_\eta^2} \in (0, 1). \quad (47)
\end{align}

If all informed traders have a trading strategy of the type $d x_t^j = \gamma_t^j \, dt + \mu_t^j \, ds_t$, denote the aggregate demand of all informed traders except $i$ by $d x_{-i}^t = \sum_{j \neq i} \gamma_t^j \, dt + \mu_t^j \, ds_t = \gamma_{-i}^t \, dt + \mu_{-i}^t \, ds_t$. Then

\begin{equation}
    d P_t = \lambda_t (\sigma_{u,t} \, du_t + (\gamma_t + \gamma_{-i}^t) \, dt + (\mu_t + \mu_{-i}^t) \, ds_t). \quad (48)
\end{equation}

Suppose informed trader $i$ also has a strategy of the form

\begin{equation}
    d x_t^i = \gamma_t \, dt + \mu_t \, ds_t. \quad (49)
\end{equation}

The expected profit of $i = 1, \ldots, N$ at $t$ is given by

\begin{align}
    \pi_{i,t}^s &= E_t^s \int_t^1 (v_1 - P_\tau - d P_\tau) \, dx_\tau^i \\
    &= E_t^s \int_t^1 (v_1 - P_\tau)(\gamma_\tau \, d \tau + \mu_\tau \, ds_\tau) - \lambda_\tau (\mu_\tau + \mu_{-i}^\tau) \mu_\tau (ds_\tau)^2 \\
    &= E_t^s \int_t^1 (v_1 - P_\tau)(\gamma_\tau \, d \tau + \mu_\tau \, ds_\tau) - \lambda_\tau (\mu_\tau + \mu_{-i}^\tau) \mu_\tau (\sigma_v^2 + \sigma_\eta^2) \, d \tau. \quad (51)
\end{align}

But $P_\tau$ is also affected by the choice of $\gamma$ and $\mu$. If $\tau > t$,

\begin{equation}
    P_\tau = P_t + \int_t^\tau \lambda_t \, dy_t = P_t + \int_t^\tau \lambda_t (\sigma_{u,t} \, du_t + (\gamma_t + \gamma_{-i}^t) \, dl + (\mu_t + \mu_{-i}^t) \, ds_t). \quad (52)
\end{equation}
For example, if $\delta_l$ is some deterministic function, define $P_r(\varepsilon)$ by replacing $\gamma_l$ with $\gamma_l + \varepsilon \delta_l$ in the above formula for $P_r$. Then

$$P'(\varepsilon) = \int_t^T \lambda_l \delta_l \, dl, \quad P''(\varepsilon) = 0. \quad (53)$$

We apply the method of calculus of variations. For some deterministic function of time, $\delta_r$, we define $f(\varepsilon)$ to be the expected profit $\pi_{i,t}^*$, with $\gamma_r$ replaced by $\gamma_r(\varepsilon) = \gamma_r + \varepsilon \delta_r$:

$$f(\varepsilon) = E\int_t^1 \left( v_1 - P_r(\varepsilon) \right)(\gamma_r(\varepsilon) \, d\tau + \mu_r \, ds_r) - \lambda_r(\mu_r + \mu^*_r)\mu_r(\sigma^2_{\mu_r} + \sigma^2_{\eta_r}) \, d\tau. \quad (54)$$

Using Equation (53), we compute

$$f'(\varepsilon) = E\int_t^1 \left( -\left( \int_t^\tau \lambda_l \delta_l \, dl \right)(\gamma_r(\varepsilon) \, d\tau + \mu_r \, ds_r) + (v_1 - P_r(\varepsilon))\delta_r \, d\tau \right), \quad (55)$$

$$f''(\varepsilon) = -2E\int_t^1 \left( \int_t^\tau \lambda_l \delta_l \, dl \right) \delta_r \, d\tau. \quad (56)$$

Define $D_r = \int_t^\tau \lambda_l \delta_l \, dl$. We obtain

$$f'(0) = E\int_t^1 \left( \frac{v_1 - P_r}{\lambda_r} D_r \right) \delta_r \, d\tau - \left( \int_t^\tau \lambda_l \delta_l \, dl \right)(\gamma_r \, d\tau + \mu_r \, ds_r))$$

$$= E\int_t^1 \left( \frac{v_1 - P_r}{\lambda_r} D_r \right) \delta_r \, d\tau - D_r(\gamma_r \, d\tau + \mu_r \, ds_r)) \right).$$

Given that $D_r$ is a deterministic function of $\tau$, we can use integration by parts

$$f'(0) = E\int_t^1 \left( \frac{v_1 - P_r}{\lambda_r} D_r \right) \delta_r \, d\tau - \int_t^1 d\left( \frac{v_1 - P_r}{\lambda_r} \right) D_r(\gamma_r \, d\tau + \mu_r \, ds_r))$$

$$= E\int_t^1 \left( \frac{v_1 - P_r}{\lambda_r} D_r \right) \delta_r \, d\tau - \int_t^1 \left( \frac{v_1 - P_r}{\lambda_r} \right) D_r(\gamma_r \, d\tau + \mu_r \, ds_r)) \right).$$
Using Equation (48), and $E_t^*(du_r) = E_t^*(ds_r) = 0$ for $\tau \geq t$, we obtain

$$f'(0) = \frac{E_t^*(v_1 - P_1)}{\lambda_1} D_1 - E_t^* \int_t^1 \left( (v_1 - P_\tau) \left( \frac{1}{\lambda_\tau} \right)' - \gamma^{-i}_\tau \right) D_\tau d\tau$$

$$= \frac{E_t^*(v_1 - P_1)}{\lambda_1} D_1 - \int_t^1 E_t^* \left( (v_1 - P_\tau) \left( \frac{1}{\lambda_\tau} \right)' - \gamma^{-i}_\tau \right) D_\tau d\tau.$$

Since the function $\delta_\tau$ is arbitrary, $D_\tau$ is also arbitrary (with $D_t = 0$). The value $D_1$ is also arbitrary. By the usual lemmas of variational calculus (see Gelfand and Fomin (2003), the first order condition for an optimum, $f'(0) = 0$ translates into

$$E_t^*(v_1 - P_1) = 0,$$  \hspace{1cm} (57)

$$E_t^* \left( (v_1 - P_\tau) \left( \frac{1}{\lambda_\tau} \right)' - \gamma^{-i}_\tau \right) = 0 \hspace{1cm} \forall \tau \geq t.$$  \hspace{1cm} (58)

Equation (58) for $\tau = t$ is

$$(w_t^s - P_t) \left( \frac{1}{\lambda_t} \right)' - \gamma^{-i}_t = 0.$$  \hspace{1cm} (59)

If $N = 1$, the residual demand $\gamma^{-i}_t = 0$. Since the filtered fundamental value $w_t^s$ is almost surely different from $P_t$, Equation (59) implies that $\lambda_t$ is constant. As in the Kyle (1985) or Back (1992) model, when $\lambda_t$ is constant, there are many strategies that give the same expected profit, as long as the condition $E_t^*(v_1 - P_1) = 0$ is satisfied.

If $N \geq 2$, Equation (59) implies that the residual demand has the form

$$\gamma^{-i}_t = \beta_t^{-i}(w_t^s - P_t) \hspace{1cm} \text{with} \hspace{1cm} \beta_t^{-i} = -\frac{\lambda'_t}{\lambda_t^2}.$$  \hspace{1cm} (60)

Since this is true for all $N$ traders, it follows that $\gamma_t = \beta_t (w_t^s - P_t)$, with $\beta_t = \frac{1}{N-1} \beta_t^{-i} = -\frac{1}{N-1} \frac{\lambda'_t}{\lambda_t^2}$, which is the desired formula (16).

To understand the condition $E_t^*(v_1 - P_1) = 0$, define

$$z_{t,\tau}^* = E_t^*(P_\tau).$$  \hspace{1cm} (61)
The infinitesimal change in $z_{t,\tau}^s$ with respect to $\tau$ is

$$z_{t,\tau+\,d\tau}^s - z_{t,\tau}^s = E_t^s(\,dP_\tau) = N\lambda_\tau \beta_\tau (w_\tau - z_{t,\tau}^s).$$

This differential equation in $\tau$ has the boundary condition $z_{t,t}^s = P_t$, so we obtain

$$w_t^s - z_{t,\tau}^s = (w_t^s - P_t) e^{-N \int_t^\tau \lambda_l \beta_l \,dl}. \quad (62)$$

The condition $E_t^s(v_1 - P_1) = 0$ translates into $e^{-N \int_t^1 \lambda_l \beta_l \,dl} = 0$, or $\int_t^1 \lambda_l \beta_l \,dl = +\infty$. This is Equation (14).

When $N = 1$, $\lambda_\tau = \lambda$ is constant, hence $f''(0) = -2 E_t \int_t^1 D_\tau \delta_\tau \,d\tau = -2 E_t \int_t^1 \frac{1}{\lambda_\tau} D_\tau D_\tau' \,d\tau = -E_t \int_t^1 \frac{1}{\lambda_\tau} (D_\tau^2)' \,d\tau = -2 E_t \left( \frac{1}{\lambda_1} D_1^2 \right) + E_t \int_t^1 \left( \frac{\lambda'_\tau}{\lambda^2_\tau} \right) D_\tau^2 \,d\tau, \quad (63)$$

where the last equality is obtained by integration by parts.

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where the last equality is obtained by integration by parts.

When $N = 1$, $\lambda_\tau = \lambda$ is constant, hence $f''(0) = -2 E_t \int_t^1 D_\tau \delta_\tau \,d\tau = -2 E_t \int_t^1 \frac{1}{\lambda_\tau} D_\tau D_\tau' \,d\tau = -E_t \int_t^1 \frac{1}{\lambda_\tau} (D_\tau^2)' \,d\tau = -2 E_t \left( \frac{1}{\lambda_1} D_1^2 \right) + E_t \int_t^1 \left( \frac{\lambda'_\tau}{\lambda^2_\tau} \right) D_\tau^2 \,d\tau.$

When $N \geq 2$, $\lambda_\tau = \lambda$ is constant, hence $f''(0) = -2 E_t \int_t^1 D_\tau \delta_\tau \,d\tau = -2 E_t \int_t^1 \frac{1}{\lambda_\tau} D_\tau D_\tau' \,d\tau = -E_t \int_t^1 \frac{1}{\lambda_\tau} (D_\tau^2)' \,d\tau = -2 E_t \left( \frac{1}{\lambda_1} D_1^2 \right) + E_t \int_t^1 \left( \frac{\lambda'_\tau}{\lambda^2_\tau} \right) D_\tau^2 \,d\tau.$
replace $\gamma_\tau$ with $\beta_\tau(w^s_\tau - P_\tau)$:

$$\pi_{i,t}^s = E_t^s \left( \int_t^{\tau} (v_1 - P_\tau)(\beta_\tau(w^s_\tau - P_\tau) \, d\tau + \mu_\tau \, ds_\tau) - (\sigma^2_v + \sigma^2_\eta) \lambda_\tau(\mu_\tau + \mu^{-1}_\tau) \mu_\tau \, d\tau \right)$$

$$= E_t^s \left( \int_t^{\tau} \beta_\tau(v_1 - P_\tau)(w^s_\tau - P_\tau) \, d\tau + \mu_\tau(v_1 - P_\tau) \, ds_\tau - (\sigma^2_v + \sigma^2_\eta) \lambda_\tau(\mu_\tau + \mu^{-1}_\tau) \mu_\tau \, d\tau \right)$$

Define

$$\Omega_{t,\tau}^s = E_t^s[(w^s_\tau - P_\tau)^2]$$

Since $w^s_\tau = E(v_1|\mathcal{F}_t^s)$, it follows that $E_t^s[(v_1 - P_\tau)(w^s_\tau - P_\tau)] = E_t^s[(w^s_\tau - P_\tau)^2] = \Omega_{t,\tau}^s$.

Also, note that $ds_\tau = dv_\tau + d\eta_\tau$ is orthogonal to $P_\tau$ and all the increments of $v$ except $dv_\tau$. This implies that $E_t^s[(v_1 - P_\tau) \, ds_\tau] = E_t^s(dv_\tau \, ds_\tau) = \sigma^2_v \, d\tau$. We finally obtain

$$\pi_{i,t}^s = \int_t^{\tau} \beta_\tau \Omega_{t,\tau}^s \, d\tau + \sigma^2_v \int_t^{\tau} \mu_\tau \, d\tau - (\sigma^2_v + \sigma^2_\eta) \int_t^{\tau} \lambda_\tau(\mu_\tau + \mu^{-1}_\tau) \mu_\tau \, d\tau$$

To compute the infinitesimal change in $\Omega_{t,\tau}^s$ with respect to $\tau$, start with

$$\Omega_{t,\tau+\delta\tau}^s = E_t^s[(w^s_\tau + dw^s_\tau - P_\tau - dP_\tau)^2]$$

$$= \Omega_{t,\tau}^s - 2E_t^s[(w^s_\tau - P_\tau)(dw^s_\tau - dP_\tau)] + E_t^s[(dw^s_\tau - dP_\tau)^2],$$

where $dw^s_\tau = a \, ds_\tau$, and $dP_\tau = \lambda_\tau(\sigma_{u,t} \, du_\tau + N\beta_\tau(w^s_\tau - P_\tau) \, dt + (\mu_\tau + \mu^{-1}_\tau) \, ds_\tau)$. We get the first order differential equation

$$\frac{d\Omega_{t,\tau}^s}{d\tau} = -2N\lambda_\tau\beta_\tau \Omega_{t,\tau}^s + \left(\lambda^2_\tau a^2_{u,\tau} + (a - \lambda_\tau(\mu_\tau + \mu^{-1}_\tau))^2(\sigma^2_v + \sigma^2_\eta)\right), \quad (66)$$

with boundary condition $\Omega_{t,t}^s = 0$. The solution is

$$\Omega_{t,\tau}^s = e^{-2NL_\tau} \int_t^{\tau} \left(\lambda^2_\tau a^2_{u,\tau} + (a - \lambda_\tau(\mu_\tau + \mu^{-1}_\tau))^2(\sigma^2_v + \sigma^2_\eta)\right) e^{2NL_{t,l}} \, dl \quad (67)$$

$$L_\tau = \int_t^{\tau} \lambda_\tau \beta_\tau \, dl. \quad (68)$$
If \( N \geq 2 \), we have \( \lambda_l \beta_l = -\frac{1}{N-1} \lambda_l^\prime \), hence \( L_\tau = \frac{1}{N-1} \log \left( \frac{N-1}{N} \right) \). Thus,

\[
\Omega_{t,\tau}^\varepsilon = (\sigma_v^2 + \sigma_\eta^2) \lambda_{\tau}^{2N-1} \int_t^\tau \frac{\lambda_l^2 \sigma_v^2 + (a - \lambda_l (\mu_l + \mu_{l^-i})^2)}{\lambda_l^{2N-1}} \, dl. \tag{69}
\]

If we denote

\[
A_\tau = \frac{\lambda_l^2 \sigma_v^2}{\sigma_v^2 + \sigma_\eta^2},
\]

we have

\[
\beta_\tau \Omega_{t,\tau}^\varepsilon = -\frac{\sigma_v^2 + \sigma_\eta^2}{N-1} \lambda_l^\prime \lambda_{\tau}^{2N-1} \int_t^\tau \frac{A_\tau + (a - \lambda_l (\mu_l + \mu_{l^-i})^2)}{\lambda_l^{2N-1}} \, dl. \tag{71}
\]

Rewrite Equation (65) and divide the expected profit by \( \sigma_v^2 + \sigma_\eta^2 \) (recall \( a = \frac{\sigma_v^2}{\sigma_v^2 + \sigma_\eta^2} \)):

\[
\frac{\pi_{i,t}^\varepsilon}{\sigma_v^2 + \sigma_\eta^2} = -\frac{1}{N-1} \int_t^1 \lambda_l^\prime \lambda_{\tau}^{2N-1} \int_t^\tau \frac{A_\tau + (a - \lambda_l (\mu_l + \mu_{l^-i})^2)}{\lambda_l^{2N-1}} \, dl \, d\tau
+ a \int_t^1 \mu_\tau \, d\tau - \int_t^1 \lambda_\tau (\mu_\tau + \mu_{l^-i}) \mu_\tau \, d\tau. \tag{72}
\]

We write \( \frac{1}{N-1} \lambda_l^\prime \lambda_{\tau}^{2N-1} = \frac{1}{N+1} \left( \lambda_{\tau}^{\frac{N+1}{2N-1}} \right)^\prime \). Integration by parts and \( \lambda_1 = 0 \) imply

\[
\frac{\pi_{i,t}^\varepsilon}{\sigma_v^2 + \sigma_\eta^2} = \frac{1}{N+1} \int_t^1 A_\tau + (a - \lambda_\tau (\mu_\tau + \mu_{l^-i}))^2 \, d\tau + a \int_t^1 \mu_\tau \, d\tau - \int_t^1 \lambda_\tau (\mu_\tau + \mu_{l^-i}) \mu_\tau \, d\tau.
\]

We define \( g(\varepsilon) \) by replacing \( \mu_\tau \) with \( \mu_\tau(\varepsilon) = \mu_\tau + \varepsilon \delta_\tau \):

\[
g(\varepsilon) = \frac{1}{N+1} \int_t^1 A_\tau + (a - \lambda_\tau (\mu_\tau(\varepsilon) + \mu_{l^-i}))^2 \, d\tau + a \int_t^1 \mu_\tau(\varepsilon) \, d\tau - \int_t^1 \lambda_\tau (\mu_\tau(\varepsilon) + \mu_{l^-i}) \mu_\tau(\varepsilon) \, d\tau.
\]

Then we compute

\[
g'(\varepsilon) = -\frac{2}{N+1} \int_t^1 \delta_\tau (a - \lambda_\tau (\mu_\tau(\varepsilon) + \mu_{l^-i})) \, d\tau + a \int_t^1 \delta_\tau \, d\tau - \int_t^1 \lambda_\tau \delta_\tau (2\mu_\tau(\varepsilon) + \mu_{l^-i}) \, d\tau.
\]
If we set $\varepsilon = 0$, we get

$$g'(0) = \int_t^1 \delta_{\tau} \left( -\frac{2(a - \lambda_{\tau}(\mu_{\tau} + \mu_{\tau}^{-i}))}{N + 1} + a - \lambda_{\tau}(2\mu_{\tau} + \mu_{\tau}^{-i}) \right) \mathrm{d}\tau. \quad (73)$$

The first order condition is $g'(0) = 0$ for any $\delta_{\tau}$, which implies

$$-\frac{2(a - \lambda_{\tau}(\mu_{\tau} + \mu_{\tau}^{-i}))}{N + 1} + a - \lambda_{\tau}(2\mu_{\tau} + \mu_{\tau}^{-i}) = 0. \quad (74)$$

Since this is true for all $N \geq 2$ informed agents, we get $-\frac{2(a - N\lambda_{\tau}\mu_{\tau})}{N + 1} + a - (N + 1)\lambda_{\tau}\mu_{\tau}$, which implies that $\mu_{\tau} = \frac{a(N-1)}{(N^2+1)\lambda_{\tau}}$. This proves Equation (17). We further compute

$$g''(\varepsilon) = \frac{2}{N + 1} \int_t^1 \lambda_{\tau}\delta_{\tau}^2 \mathrm{d}\tau - 2 \int_t^1 \lambda_{\tau}\delta_{\tau}^2 \mathrm{d}\tau = -\frac{2N}{N + 1} \int_t^1 \lambda_{\tau}\delta_{\tau}^2 \mathrm{d}\tau. \quad (75)$$

This proves that $g''(0) < 0$ with strict inequality, hence Equation (17) is also a sufficient condition for a local maximum.

If $N = 1$, Equation (65) becomes (after dividing by $\sigma_v^2 + \sigma_\eta^2$):

$$\frac{\pi^{*\tau}_{i,t}}{\sigma_v^2 + \sigma_\eta^2} = \int_t^1 \beta_{\tau} e^{-2L_{\tau}} \left( \int_t^\tau (A_{t'} + (a - \lambda_{t'}(\mu_{t'} + \mu_{t'}^{-i}))^2) e^{2L_{t'}} \mathrm{d}l \right) \mathrm{d}\tau$$

$$+ a \int_t^1 \lambda_{\tau} \mathrm{d}\tau - \int_t^1 \lambda_{\tau}(\mu_{\tau} + \mu_{\tau}^{-i}) \mu_{\tau} \mathrm{d}\tau. \quad (76)$$

Because $\lambda$ is constant, we can write $\beta_{\tau} e^{-2L_{\tau}} = -\frac{1}{2\lambda} (e^{-2L_{\tau}})'$. The condition $\int_t^1 \beta_{\tau} \mathrm{d}l = +\infty$ translates into $e^{-2L_{\tau}} = 0$. We integrate by parts in the above formula, and obtain

$$\frac{\pi^{*\tau}_{i,t}}{\sigma_v^2 + \sigma_\eta^2} = \frac{1}{2\lambda} \int_t^1 A_{t'} + (a - \lambda(\mu_{t'} + \mu_{t'}^{-i}))^2 \mathrm{d}\tau + a \int_t^1 \lambda_{\tau} \mathrm{d}\tau - \lambda \int_t^1 (\mu_{t'} + \mu_{t'}^{-i}) \mu_{\tau} \mathrm{d}\tau.$$

If, as before, we define $g(\varepsilon)$ as $\frac{\pi^{*\tau}_{i,t}}{\sigma_v^2 + \sigma_\eta^2}$ where $\mu_{\tau}$ is replaced by $\mu_{\tau} + \varepsilon$, we compute

$$g'(0) = -\int_t^1 \delta_{\tau}(a - \lambda(\mu_{\tau} + \mu_{\tau}^{-i})) \mathrm{d}\tau + a \int_t^1 \delta_{\tau} \mathrm{d}\tau - \lambda \int_t^1 \delta_{\tau}(2\mu_{\tau} + \mu_{\tau}^{-i}) \mathrm{d}\tau$$

$$= \int_t^1 \delta_{\tau} \left( -a + \lambda(\mu_{\tau} + \mu_{\tau}^{-i}) + a - \lambda(2\mu_{\tau} + \mu_{\tau}^{-i}) \right) \mathrm{d}\tau$$

$$= -\lambda \int_t^1 \delta_{\tau}\mu_{\tau} \mathrm{d}\tau. \quad (77)$$
Since $g'(0) = 0$ for any $\delta_r$, it follows that, when $N = 1$, indeed $\mu_r = 0$. This is a necessary condition for a local maximum. To show that the condition is also sufficient, we note that Equation (75) is valid when $N = 1$, hence $g''(0) < 0$, i.e., the second order condition is satisfied at $\mu_r = 0$.

Next, for any $N$, we use price efficiency to compute $\lambda_t$. Start from $P_t = \mathbb{E}(v_t | \mathcal{F}_t) = \mathbb{E}_t(v_1)$, where we use the subscript $t$ to indicate conditioning on all public information available at $t$. Then $P_{t+dt} = \mathbb{E}(v_t | \mathcal{F}_t, dy_t)$, which implies

$$dP_t = \mathbb{E}_t(v_1 - P_t | dy_t) = \frac{\text{cov}_t(v_1 - P_t, dy_t)}{\text{var}_t(dy_t)} dy_t$$

$$= \frac{N\beta t \text{cov}_t(v_1, w^*_t) dt + N\mu t \text{cov}(v_1, ds_t)}{\sigma^2_{u,t} dt + N^2\mu_t^2(\sigma^2_v + \sigma^2_\eta) dt} dy_t$$

$$= \frac{N\beta t \Sigma_t + N\mu t \sigma^2_v}{\sigma^2_{u,t} + N^2\mu_t^2(\sigma^2_v + \sigma^2_\eta)} dy_t$$

$$= \lambda_t dy_t.$$  

This implies $\lambda_t = \frac{N\beta t \Sigma_t + N\mu t \sigma^2_v}{\sigma^2_{u,t} + N^2\mu_t^2(\sigma^2_v + \sigma^2_\eta)}$, or $\sigma^2_{u,t} \lambda_t^2 + (N\mu t \lambda_t)^2(\sigma^2_v + \sigma^2_\eta) = N\beta t \lambda_t \Sigma_t + N\mu t \lambda_t \sigma^2_v$. To simplify notation, define the following normalized variables:

$$c_t = \frac{\sigma^2_{u,t}}{\sigma^2_v + \sigma^2_\eta}, \quad \hat{\Sigma}_t = \frac{\Sigma_t}{\sigma^2_v + \sigma^2_\eta}. \quad (78)$$

The equation for $\lambda_t$ can be rewritten as $c_t + (N\mu t \lambda_t)^2 = N\beta t \lambda_t \Sigma_t + N\mu t \lambda_t a$. From (17), we get $N\mu t \lambda_t = av$, therefore $c_t \lambda_t^2 + a^2v^2 = N\beta t \lambda_t \Sigma_t + a^2v$, or $N\beta t \lambda_t \Sigma_t = c_t \lambda_t^2 - a^2(\nu - v^2)$, which after multiplying by $\sigma^2_v + \sigma^2_\eta$ is the same as (18).

Next, we compute $\Sigma_t = \text{Var}_t(w^*_t) = \text{Var}(w^*_t - P_t) = \mathbb{E}((w^*_t - P_t)^2)$. Consider $\Sigma_{t+dt} = \mathbb{E}((w^*_{t+dt} - P_{t+dt})^2)$. After a similar computation to that for $\Omega^*_t$, we obtain the first order differential equation for $\Sigma_t$:

$$\frac{d\Sigma_t}{dt} = -2N\lambda_t \beta_t \Sigma_t + \left(\lambda_t^2 \sigma^2_{u,t} + (a - N\lambda_t \mu_t)^2(\sigma^2_v + \sigma^2_\eta)\right), \quad (79)$$

with $\Sigma_0$ equal to the variance of the initial filtered value of the informed traders. Using normalized values, this is the same as $\hat{\Sigma}_t = -2N\lambda_t \beta_t \hat{\Sigma}_t + c_t \lambda_t^2 + (a - N\lambda_t \mu_t)^2$. From (18),
\( N \beta_t \lambda_t \Sigma_t = c_t \lambda_t^2 - a^2(\nu - \nu^2) \), so we get \( \Sigma_t' = -2c_t \lambda_t^2 + 2a^2(\nu - \nu^2) + c_t \lambda_t^2 + a^2(1 - \nu)^2 = a^2(1 - \nu^2) - c_t \lambda_t^2 \). This is equivalent to (19).

Finally, we prove Equation (20). Denote \( X_t = \text{Var}(v_1 - w_t^1) \). Compute \( X_{t+dt} = \text{Var}(v_1 - w_{t+dt}^1) = \text{Var}(v_1 - w_t^1 - a \, ds_t) = X_t + a^2(\sigma_1^2 + \sigma_2^2) \, dt - 2a \, \text{Cov}(v_1 - w_t^1, \, ds_t) = X_t - a \sigma_2^2 \, dt \). Thus, \( X_t' = -a \sigma_2^2 \), which implies \( X_t = X_0 - a \sigma_2^2 t \). In particular, \( X_1 - X_0 = -a \sigma_2^2 \). Next, consider \( V_t = \text{Var}(v_1 | \mathcal{F}_t) \). Then \( V_t = \text{Var}(v_1 - w_t^1 + w_t^1 | \mathcal{F}_t) = \text{Var}(v_1 - w_t^1) + \text{Var}(w_t^1 | \mathcal{F}_t) \), since \( v_1 - w_t^1 \) is orthogonal to \( \mathcal{F}_t^s \) (and hence also to \( \mathcal{F}_t \)). This implies that \( V_t = X_t + \Sigma_t \).

We also have \( V_t = \text{Var}(v_1 - P_t + P_t | \mathcal{F}_t) = \text{Var}(v_1 - P_t) \), since \( P_t \in \mathcal{F}_t \), and \( v_1 - P_t \) is orthogonal to \( \mathcal{F}_t \). We write \( v_1 - P_t = \frac{\text{Cov}(v_1 - P_t, dy_t)}{\text{Var}(dy_t)} \, dy_t + \varepsilon = \lambda_t \, dy_t + \varepsilon \), where \( \varepsilon \) is orthogonal to both \( \mathcal{F}_t \) and \( dy_t \), the change in aggregate demand. Taking variances on both sides, we get \( V_t = \lambda_t^2(\sigma_{t,1}^2 + N^2 \mu_t^2(\sigma_{t,1}^2 + \sigma_{t,2}^2)) \, dt + V_{t+dt} \). This implies \( V_t' = -\left(\sigma_{t,1,2}^2 \lambda_t^2 + a \sigma_2^2 \nu^2\right) \). Integrating over \( t \), we get \( V_0 - V_1 = \int_0^1 \left(\sigma_{t,1,2}^2 \lambda_t^2 + a \sigma_2^2 \nu^2\right) \, dt \). But \( V_t = X_t + \Sigma_t \), with \( X_0 - X_1 = a \sigma_2^2 \), and, as we show below, \( \Sigma_1 = 0 \). These combined imply that \( \int_0^1 \left(\sigma_{t,1,2}^2 \lambda_t^2 + a \sigma_2^2 \nu^2\right) \, dt = \Sigma_0 - \Sigma_1 + a \sigma_2^2 \). In order to prove (20), we only have to show that \( \Sigma_1 = 0 \).

If \( N = 1 \), \( \lambda_t \) is constant, and \( \nu = 0 \). Equation (18) implies \( \Sigma_t = \frac{\lambda \sigma_2^2}{\beta_t} \). But condition (14) implies that \( \beta_1 \to 0 \) when \( t \to 1 \). Thus, as long as \( \sigma_{t,1,2} \) is well behaved when \( t \to 1 \), we have \( \Sigma_1 \to 0 \).

If \( N \geq 2 \), \( \beta_t = -\frac{1}{N-1} \frac{\lambda_t}{\lambda_t^2} \), hence Equation (18) implies \( \Sigma_t = \frac{\sigma_{t,1,2}^2 \lambda_t^2 - a \sigma_2^2 (\nu - \nu^2) \lambda_t}{N-1} \). Condition (15) says that \( \lambda_1 \to 0 \) when \( t \to 1 \). When \( \lambda_t \) is well behaved near 1 (for example if it has a Taylor series in \( 1 - t \)), then it converges faster to zero than \( \lambda_t \). Thus, \( \Sigma_1 \to 0 \) when \( t \to 1 \).

Finally, if \( N \geq 2 \), consider the formula (51) for the expected profit of informed trader \( i \). We consider a deviation \( h(\varepsilon^1, \varepsilon^2) = (\gamma \tau + \varepsilon^1 \delta \tau + \mu \, \tau + \varepsilon^2 \delta \tau \) from the strategy \( (\gamma \tau, \mu \tau) \).

Then consider the Jacobian matrix of \( h \) with respect to \( (\varepsilon^1, \varepsilon^2) \), and compute it at \( (0, 0) \):

\[
J(0, 0) = \begin{bmatrix}
-2 \int_t^1 \delta_t^1 \left(f_t^l \lambda_0 \delta^1_t dl\right) \, d\tau & -\int_t^1 \delta_t^1 \left(f_t^l \lambda_0 \delta^2_t dl\right) \, d\tau \\
-\int_t^1 \delta_t^1 \left(f_t^l \lambda_0 \delta^2_t dl\right) \, d\tau & -2 \int_t^1 \lambda_t (\delta^2_t)^2 \, d\tau
\end{bmatrix}.
\]

One can show that the Jacobian matrix is strictly negative definite for all functions
\( \delta^1_r \) and \( \delta^2_r \), and thus the expected profit is a concave functional. Therefore, the local maximum given by the solution is also global.

**Proof of Proposition 1:** When \( N \) is large, \( \nu = N^2 - N \approx 1. \) Also, \( N \beta_t \lambda_t = -N \frac{\lambda_t}{N-1} \approx -\frac{\lambda_t}{N} \). Equation (21) becomes \( \left( \frac{\sigma^2 \lambda^3}{N} \right)' = \sigma^2 \lambda^2 \). This implies \( \frac{2\sigma_u \sigma_v^2 \lambda_3 + \sigma^2 \lambda^3 \lambda''}{(\lambda^2)'} = \sigma^2 \lambda^2 \). Multiply this by \( \frac{\lambda'}{N \sigma^2} \). We obtain \( \frac{2\lambda''}{\sigma_u} + 3 \frac{\lambda'}{\lambda} = \frac{\lambda'}{\lambda} \). This is equivalent to (22). The equation can be rewritten as \((\ln(\lambda'))' = (2 \ln(\lambda) + 2 \ln(\sigma_u))'\), which implies \( \ln(\lambda') = \ln(\sigma^2_u \lambda^2) + C_0 \). Taking the exponential of this, it follows that \( \lambda' = -C \sigma^2_u \lambda^2 \) for some constant \( C \). Thus, \( \left( \frac{1}{\lambda} \right)' = C \sigma^2_u \), which implies \( \frac{1}{\lambda} = \frac{1}{\lambda_0} + C \int_0^1 \sigma^2_u \, d\tau \). This is Equation (24). Theorem 1 also implies that \( \int_0^1 \sigma^2_u \lambda^2 \, dt = \Sigma_0 \) (since \( \nu = 1 \)), if we substitute the formula for \( \lambda_t \), we get Equation (25).

In the particular case when \( \sigma_{u,t} \) satisfies (26), we get \( \frac{1}{\lambda_t} = \frac{1}{\lambda_0} + \frac{C(1-t)^{2-\alpha}}{2-\alpha} - \frac{C}{2-\alpha}. \) This is satisfied if \( \lambda_t = \frac{2\alpha-1}{C} (1-t)^{2-\alpha} \). Substituting this into \( \int_0^1 \sigma^2_{u,t} \lambda^2 \, dt = \Sigma_0 \), we get \( C = \frac{\sigma_{u,0}(2\alpha-1)^{1/2}}{\Sigma_0^{1/2}} \), which implies that \( \lambda_t \) satisfies (27).

**Proof of Proposition 2:** We search for a particular solution to Equation (21) of the form \( \lambda_t = C^{1/2} (1-t)^b \). If we denote by \( D = \sigma^2_u,0 \), we know that \( \sigma^2_u,t = D(1-t)^{-2\alpha} \).

With this notation, Equation (20) becomes \( \int_0^1 CD(1-t)^{2(\alpha-\nu)} \, dt = \Sigma_0 + a\sigma^2_v (1-\nu^2) \), therefore \( \frac{CD}{2(\alpha-\nu)+1} = \Sigma_0 + a\sigma^2_v (1-\nu^2) \). Hence, we get \( C = \frac{(\Sigma_0 + a\sigma^2_v (1-\nu^2))}{\left(2(\alpha-\nu)+1\right)} \).

Equation (18) implies \( \Sigma_t = \frac{\sigma^2_{u,t} \lambda^2 - a^2 (\nu - \nu^2)}{N-1} \), which implies \( \Sigma_t = \left( CD(1-t)^{2(\alpha-\nu^2)} - a\sigma^2_v (1-\nu^2) \right) \frac{N-1-t}{N} = \frac{N-1}{N} CD(1-t)^{2(\alpha-\nu)+1} - \frac{N-1}{N} a\sigma^2_v (1-\nu^2) (1-t) \). Differentiate this with respect to \( t \), to get \( \Sigma'_t = \frac{N-1}{N} a\sigma^2_v (\nu - \nu^2) N(2(\alpha-\nu)+1) (1-t) \).

But Equation (19) implies \( \Sigma'_t = a\sigma^2_v (1-\nu^2) - CD(1-t)^{2(\alpha-\nu)} \). If we require that the two formulas for \( \Sigma'_t \) are the same, we get the following system in \( \alpha \) and \( b \):

\[
\frac{N-1}{Nb} (\nu - \nu^2) = 1 - \nu^2, \quad \tag{81}
\]

\[
\frac{N-1}{Nb} (2(\alpha-\nu)+1) = 1. \tag{82}
\]

Since \( \nu = \frac{N^2 - N}{N^2 + 1} \), the solution of the system is given by \( \alpha \) as in (29) and \( b \) as in (33).

Equation (82) also implies that \( 2(\alpha-\nu)+1 = \frac{bN}{N-1} \). Using the above formula for \( C \), we compute \( C = \left( \Sigma_0 + a\sigma^2_v (1-\nu^2) \right) (2(\alpha-\nu)+1) = \frac{\Sigma_0 + a\sigma^2_v (1-\nu^2)}{\sigma^2_{u,0}} \frac{bN}{N-1} \). This implies
Equation (30) for $\lambda_t = C^{1/2}(1-t)^b$. Equation (31) is given by direct computation, using the formula (16), $\beta_t = -\frac{1}{N-1} \frac{\lambda_t}{\lambda'_t}$. Equation (32) is simply (17). \[ \square \]

**Proof of Theorem 2:** If $A(\lambda) = \sigma^2_u(\lambda)\lambda^2$, Equation (21) becomes \( \left( \frac{A-a\sigma^2_v(\nu-\nu^2)}{N-1} \right)' = a\sigma^2_v(1-\nu^2) - A \). After some algebraic manipulation, this can be rewritten as

\[
\frac{A'\lambda + \frac{1}{N} A + a\sigma^2_v \left( \frac{N-1}{N} (1-\nu^2) - (\nu-\nu^2) \right)}{\lambda(A - a\sigma^2_v(\nu-\nu^2))} \lambda' = \frac{\lambda''}{\lambda}.
\]

Note that condition (34) implies that $A(\lambda) > a\sigma^2_v(1-\nu^2) > a\sigma^2_v(\nu-\nu^2)$, therefore the denominator in the equation above is positive. Given the definition (36) of $F(\lambda)$, we get $F'(\lambda)\lambda' = \frac{\lambda''}{\lambda}$, or $(F(\lambda))' = (\ln(\lambda))'$. This implies $\ln(\lambda') = F(\lambda) + C_0$ for some constant $C_0$, or $\lambda' = -Ce^{F(\lambda)}$ for some constant $C$. Then $\lambda' e^{-F(\lambda)} = -C$, or $(\int_0^{\lambda_t} e^{-F(x)} \, dx)' = -C$, which implies $\int_0^{\lambda_t} e^{-F(x)} \, dx = D - Ct$ for some constant $D$. Condition (15) implies that $\lambda_1 = 0$, hence if we set $t = 1$, we have $0 = D - C$, or $D = C$. We therefore get $\int_0^{\lambda_t} e^{-F(x)} \, dx = C(1-t)$, which is (35). The constant $C$ is determined by $\int_0^1 A(\lambda_t) \, dt = \Sigma_0 + a\sigma^2_v(1-\nu^2)$, which is just a rewriting of (20). The technical condition that there exist $\lambda_1, \lambda_2 > 0$ such that $A(\lambda_1) \leq \Sigma_0 + a\sigma^2_v(1-\nu^2) \leq A(\lambda_2)$ makes sure that there exists a solution to this equation.

It remains to show that $\int_0^{\lambda_t} e^{-F(x)} \, dx = C(1-t)$ has a well defined left hand side and always has a solution, if condition (34) is satisfied. If we denote $b = a\sigma^2_v(\nu-\nu^2)$, we can write

\[
F(\lambda) = \int_{\lambda_0}^{\lambda} \frac{A'(x)x + \frac{1}{N}(A(x) - b) + a\sigma^2_v \frac{N-1}{N} (1-\nu)}{x(A(x) - b)} \, dx,
\]

\[
= \int_{\lambda_0}^{\lambda} \frac{A'(x)}{A(x) - b} + \frac{1}{N x} + \frac{a\sigma^2_v \frac{N-1}{N} (1-\nu)}{x(A(x) - b)} \, dx,
\]

\[
= \ln \left( \frac{A(\lambda_0) - b}{A(\lambda) - b} \right) + \frac{\lambda_0}{N} + \frac{(N-1)b}{N\nu} \int_{\lambda_0}^{\lambda} \frac{dx}{x(A(x) - b)}.
\]

This implies that

\[
e^{-F(\lambda)} = \frac{A(\lambda_0) - b}{A(\lambda) - b} \left( \frac{\lambda_0}{\lambda} \right)^{1/N} e^{\frac{(N-1)b}{N\nu} \int_{\lambda_0}^{\lambda} \frac{dx}{x(A(x) - b)}}.
\] (83)

Condition (34) implies that $A(\lambda) - b > a\sigma^2_v(1-\nu^2) - b = a\sigma^2_v(1-\nu)$. If we denote
\[ B = a\sigma_v^2(1 - \nu), \] we have \( A(\lambda) - b > B, \) which implies \( A(\lambda) - b > cB \) for some \( c > 1, \) or equivalently \( \frac{1}{A(\lambda) - b} < \frac{1}{cB}. \) It follows that for all \( \lambda \in (0, \lambda_0], \) we have \( e^{-F(\lambda)} < \frac{A(\lambda_0) - b}{A(\lambda) - b} \left( \frac{\lambda_0}{\lambda} \right)^{1/N} \frac{e^{\frac{(N-1)b}{N\nu}}}{\frac{(N-1)b}{N\nu} + \frac{b}{\gamma}}, \) or \( e^{-F(\lambda)} < \frac{A(\lambda_0) - b}{A(\lambda) - b} \left( \frac{\lambda_0}{\lambda} \right)^{1/N} \frac{1}{\frac{(N-1)b}{N\nu} + \frac{b}{\gamma}}. \) Denote by \( d = \frac{1}{N} + \frac{(N-1)b}{N\nu} \). Since \( \frac{1}{N} + \frac{(N-1)b}{N\nu} = 1 \) and \( c > 1, \) it follows that \( d < 1. \) We also have 

\[ e^{-F(\lambda)} > \frac{A(\lambda_0) - b}{cB} \left( \frac{\lambda_0}{\lambda} \right)^{1/N}, \] therefore 

\[ \frac{A(\lambda_0) - b}{A(\lambda) - b} \left( \frac{\lambda_0}{\lambda} \right)^{1/N} < e^{-F(\lambda)} < \frac{A(\lambda_0) - b}{cB} \left( \frac{\lambda_0}{\lambda} \right)^{d}. \]

The inequality \( e^{-F(\lambda)} < \frac{A(\lambda_0) - b}{cB} \left( \frac{\lambda_0}{\lambda} \right)^{d} \) with \( d < 1 \) implies that \( \int_0^\lambda e^{-F(x)} \, dx \) is well defined. The inequality \( e^{-F(\lambda)} > \frac{A(\lambda_0) - b}{A(\lambda) - b} \left( \frac{\lambda_0}{\lambda} \right)^{1/N}, \) along with the technical condition that \( A(\lambda) \) grows at most as \( \lambda^{\frac{N}{N-1}} \) when \( \lambda \to \infty \) implies that when \( \lambda \) is large enough, \( e^{-F(\lambda)} > \frac{D}{\lambda} \), for some constant \( D > 0. \) This implies that \( \int_0^\infty e^{-F(x)} \, dx = +\infty, \) which means that there exists a solution for \( \int_0^\lambda e^{-F(x)} \, dx = C(1-t) \) for all \( C > 0. \)

**Proof of Corollary 1:** If \( A(\lambda) = \Sigma_0 + a\sigma_v^2(1 - \nu^2) \) is constant, the technical conditions of the previous Theorem are satisfied, and, using Equation (83), we compute

\[ e^{-F(\lambda)} = \left( \frac{\lambda_0}{\lambda} \right)^{c}, \quad (84) \]

where \( c = \frac{1}{N} + \frac{N-1}{N} \frac{1}{1 + \frac{\lambda_0}{a\sigma_v^2(1-\nu)}} < 1. \) Then \( \int_0^\lambda e^{-F(x)} \, dx = C(1-t) \) translates into

\[ \lambda_0^{\frac{1}{1-c}} C(1-t) = C(1-t). \] Setting \( t = 0, \) we get \( \lambda_0 = C(1-c). \) Finally, we obtain that \( \lambda_t = C(1-c)(1-t)^{1-\epsilon}. \) This represents a family of equilibria parametrized by the constant \( C > 0. \)

To compute the informed variance ratio, use Equation (17), which implies \( \frac{N}{\mu_t}\lambda_t = a\nu. \) Therefore, \( \rho_t = \frac{\frac{a^2\nu^2}{\sigma_{u,t}^2 + \alpha^2\nu^2}}{\frac{a^2\nu^2}{\sigma_{u,t}^2 + \alpha^2\nu^2} + \frac{a^2\nu^2}{\sigma_{u,t}^2 + \alpha^2\nu^2}} = \frac{\frac{a^2\nu^2}{\sigma_{u,t}^2 + \alpha^2\nu^2}}{\frac{a^2\nu^2}{\sigma_{u,t}^2 + \alpha^2\nu^2} + \frac{a^2\nu^2}{\sigma_{u,t}^2 + \alpha^2\nu^2}} = \frac{\sigma_{u,t}^2}{\sigma_{u,t}^2 + \alpha^2\nu^2}. \) This is Equation (40). \[ \square \]

**Proof of Proposition 3:** When \( N = 1, \) according to Theorem 1, \( \lambda_t = \lambda \) is constant. Also, \( \nu = \frac{N^2-N}{N^2+1} = 0. \) Then Equation (18) becomes \( \Sigma_t = \frac{\sigma_{u,t}^2\lambda^2}{\beta_t \lambda} = \frac{\sigma_{u,t}^2\lambda^2}{\beta_t \lambda}. \) Equation (19) becomes \( \sigma_t = a\sigma_v^2 - \sigma_{u,t}^2\lambda^2. \) Putting the two equations together, we get \( \frac{(\sigma_{u,t}^2\lambda^2)}{\beta_t} = a\sigma_v^2 - \sigma_{u,t}^2\lambda^2. \) If we denote by \( C_t = \int_0^t \sigma_{u,t}^2 \, d\tau, \) by integration we obtain \( \frac{\sigma_{u,t}^2\lambda^2}{\beta_t} = a\sigma_v^2 - \lambda^2 C_t + D, \)

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for some constant $D$. This implies $\beta_t = \frac{\sigma^2_{u,t} \lambda}{a \sigma^2_{u,t} - \lambda^2 C_t + D}$. Setting $t = 0$, we get $\beta_0 = \frac{\sigma^2_{u,0} \lambda}{D}$.

But from the equation above for $\Sigma_t$, $\Sigma_0 = \frac{\sigma^2_{u,0} \lambda}{\beta_0} = D$. This implies $D = \Sigma_0$, and $\beta_t = \frac{\sigma^2_{u,t} \lambda}{a \sigma^2_{u,t} - \lambda^2 C_t + \Sigma_0}$.

We also need to satisfy (20), which in this case is $\int_0^1 \sigma^2_{u,t} \lambda^2 dt = a \sigma^2_v + \Sigma_0$. But $\int_0^1 \sigma^2_{u,t} dt = C_1$, hence $C_1 \lambda^2 = a \sigma^2_v + \Sigma_0$. Therefore, $\lambda^2 = \frac{a \sigma^2_v + \Sigma_0}{C_1}$, which implies (41). We can also substitute $\Sigma_0 = C_1 \lambda^2 - a \sigma^2_v$ in the equation for $\beta_t$, to get

$$\beta_t = \frac{\sigma^2_{u,t} \lambda}{\lambda^2 (C_1 - C_t) - a \sigma^2_v (1 - t)} = \frac{1}{\lambda (1 - t)} \frac{\sigma^2_{u,t}}{C_1 - C_t} - \frac{a \sigma^2_v}{\lambda^2} = \frac{1}{\lambda (1 - t)} \frac{C_1 - C_t}{1 - t} - \frac{a \sigma^2_v}{a \sigma^2_v + \Sigma_0} C_1,$$

which is the same as (42). Note that the technical condition ensures that the denominator in the formula above is positive.

Formula (43) follows from Theorem 1, Equation (17).

References


